DIFFERENTIAL FORMS

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Abstract—In this paper, we have defined polylinear form and some specific polylinear forms such as symmetric and antisymmetric. The theorem that provides a condition when some p-form is antisymmetric. Differential p-form is defined, it is proved that it can unambiguously presented as

\[
o(x) = \sum a_{ij_2...j_p}(x) (\xi_{j_2})A \ldots A(\xi_{j_p})
\]

Where \(o_{ij_2...j_p}: D \rightarrow R\), \(D\) is an open set of space \(R^m\). This differential is one n times differential when the function \(o_{ij_2...j_p}\) is also n times differential. Then, the formula for the calculation of integral of the form \(o\) on multiple \(M\) is given.

Keywords—permutation, polylinear forms, external product, differential forms.

I. ANTISYMMETRIC POLYLINEAR FORMS

PERMUTATIONS: Let’s say that \(J\) is a finite set of \(m\) elements. Each bijective reflection \(\sigma: J \rightarrow J\) is called permutation of the set \(J\).

Let’s say that \(b_1, b_2, \ldots, b_m\) are the elements of a set \(J\) and \(P_m\) is a number of its permutations. It is obvious that when \(m=1\) then \(P_m=1\). If \(m=2\), then \(P_m=2\) \(b_1, b_2; b_2, b_1\). If \(m=3\) then \(P_m=6\) \(b_1, b_2, b_3; b_1, b_3, b_2; b_2, b_1, b_3; b_2, b_3, b_1; b_3, b_1, b_2; b_3, b_2, b_1\).

By mathematical induction, it is shown that \(P_m = m!\)

If in permutation \(m_1, m_2, \ldots, m_n\) there is such a pair \((m_i, m_j)\), where \(j<i\) and \(m_i < m_j\), then we can say that pair forms INVERSION.

If the number of inversions is even, then permutation is called EVEN, if it is odd, the permutation is also called ODD.

Theorem 1: if two elements in permutation change places \(m_i, m_j\) or the parity of permutation changes.

Proof: Let’s assume that two adjacent elements of permutation change places. If the mappings \(\sigma: J_m \rightarrow J_m, \tau: J_m \rightarrow J_m\) of permutation are such that

\[
\sigma(J_m) = \{m_1, \ldots, m_{k-1}, m_k, m_{k+1}, \ldots, m_m\}
\]

\[
\tau(J_m) = \{m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_m, m_k\}
\]

Pairs \((m_i, m_j)\) in both cases simultaneously form or do not form inversion, if \(i<k+1\) and \(j>i\) are arbitrary and if \(j=k+1\) and \(i<j\) are arbitrary.

One pair that does not satisfy the legality shown is \((m_k, m_{k+1})\) that transfers in \((m_{k+1}, m_k)\) and therefore, number of inversions is changed for one, which means that parity is changed.

Let’s assume now that two arbitrary elements change places, \(m_i, m_j\) and it is required to make two \(i-j-1\) change of adjacent elements. If the number \(2 \cdot i-j-1\) is odd then the parity of permutation is changed. This proves the theorem.

Let’s take two permutations of the set \(J\)

\[
b_1 = a_1(\sigma_{b_1}), b_1, \tau = \tau(b_1), I = 1, 2, \ldots, m.
\]

Permutation products \(\sigma\) and \(\tau\) is called the permutation \(\tau') = (\sigma - \tau)(b_1), I = 1, 2, \ldots, m\), which is obtained first by the application of \(\tau\) permutation and then \(\sigma\) permutation.

Permutation \(\sigma^1\) is called REVERSE permutation \(\sigma\), if \(\sigma^1 = \sigma \cdot \sigma^1 = I\) where \(I\) is identical permutation, i.e., \(b_i = I(b_i), i = 1, 2, \ldots, m\).

If \(\sigma, \tau\) and \(\gamma\) are three arbitrary permutations then \((\sigma \cdot \tau) \cdot \gamma = \sigma \cdot (\tau \cdot \gamma)\).

In general case, product of permutation that is not a commutative operation i.e. \(\sigma \cdot \tau \neq \tau \cdot \sigma\).

Permutation \(i \rightarrow \sigma(i), \sigma(i) = \sigma_i, i \in J_m\) is called TRANSPOSITION, if there is such a pair of different numbers \(k \in J_m, l \in J_m\) that \(k = \sigma_i, l = \sigma_k\) and \(i = \sigma_i\) for any \(i\) different than \(k\) and \(l\).

If \(\sigma\) is the transposition, then \(\sigma^2\) is identical to permutation.

Mapping

\[G \rightarrow \{+1, -1\} : \sigma \rightarrow \epsilon = +1, \text{ if } \sigma \text{ even permutation} \]

\[-1, \text{ if } \sigma \text{ odd permutation}\]

That is based by SIGNATURE OF PERMUTATIONS.

II. POLYLINEAR ANTI-SYNTHETIC FORMS

Let’s say that \(E_1, E_2, \ldots, E_p\) are vector spaces over the field \(R\). Polylinear form is the mapping \(\varphi: E_1 \times E_2 \times \cdots \times E_p \rightarrow R\), so \(\forall k \in 1, 2, \ldots, p\) is valid also for the fixed element system \(a_i \in E_i, i \neq k\) function:

\[X_k \rightarrow \varphi(a_1, \ldots, a_{k-1}, x_k, a_{k+1}, \ldots, a_p)\]

meets the following conditions:

\[
\varphi(a_1, \ldots, a_{k-1}, \lambda x_k, a_{k+1}, \ldots, a_p) = \lambda \varphi(a_1, \ldots, a_{k-1}, x_k, a_{k+1}, \ldots, a_p)
\]

\[
\varphi(a_1, \ldots, a_{k-1}, x_k + x', a_{k+1}, \ldots, a_p) = \varphi(a_1, \ldots, a_{k-1}, x_k, a_{k+1}, \ldots, a_p) + \varphi(a_1, \ldots, a_{k-1}, x', a_{k+1}, \ldots, a_p).
\]
Polylinear form is anti-symmetric if it changes the sign in permutation of its two arguments.

Let’s say that

\[ E_1 \sigma E_2 = \ldots = E_{p} = R^m = E, \quad R^m \times R^m \times \ldots \times R^m = E^p, \]

Then polylinear form \( \varphi : E^p \rightarrow R \) is called p-form or polylinear form of p degree. With the help of \( \sigma J_p \rightarrow J_p \) can be defined the mapping of \( \sigma \varphi \): \n
\[ E^p \rightarrow R : \sigma \varphi (x_1, x_2, \ldots, x_p) = \varphi (x_{s_1}, x_{s_2}, \ldots, x_{s_m}) \]

Polylinear form \( \varphi : E^p \rightarrow R \) is called ANTI-SYNTHETIC, if \( \sigma \varphi = \varepsilon_\sigma \varphi, \forall \sigma \in G_m \).

It follows that \( \sigma \varphi := \varphi \) if \( \sigma \) is transposition.

By symmetrization \( S \varphi \) of the p-form of \( \varphi : E^p \rightarrow R \), we imply p-form determined by the following equation

\[ S \varphi = \sigma \varepsilon_\sigma \varphi. \]

By anti-symmetrization \( A \varphi \) of the p-form of \( \varphi : E^p \rightarrow R \), we imply p-form determined by the equation

\[ A \varphi = \frac{1}{p!} \sigma \varepsilon_\sigma \varphi. \]

**Theorem 2**: let’s say that \( E = R^m, e_1, e_2, \ldots, e_m \) is the base of \( R^m \) space. Then, p-linear form \( \varphi : E^p \rightarrow R \) is anti-symmetric, then and only then when it can be presented as follows

\[ \varphi (x_1, x_2, \ldots, x_p) = \sum \Delta_{j_1j_2 \ldots j_p} a_{j_1j_2 \ldots j_p}, \quad 1 \leq j_1 < j_2 < \ldots < j_p \leq m \]

where

\[ a_{j_1j_2 \ldots j_p} \in R, \quad \Delta_{j_1j_2 \ldots j_p} = \varepsilon_{(j_1j_2 \ldots j_p)}, \quad 1 \leq j_1 \leq p, 1 \leq k \leq m. \]

One may see that last relation is unique and \( a_{j_1j_2 \ldots j_p} = \varphi (e_{j_1j_2 \ldots j_p}) \).

Elements \( \Delta_{j_1j_2 \ldots j_p} \) for the base of space \( A_p (R^m, R) \) since the space \( A_p (R^m, R) \) has a dimension \( C_p^m = \frac{m!}{p!} \).

**Proof**: let’s say that \( E^p \rightarrow R \) is a p-linear anti-symmetric form. Then, if \( e_1, e_2, \ldots, e_m \) is the base of space \( R^m \), then \( x_i = \sum_{j=1}^{m} x_{i_j} e_j, \quad i = 1, 2, \ldots, p \), and due to polilinearity \( \varphi \)

\[ \varphi (x_1, x_2, \ldots, x_p) = \sum_{j_1j_2 \ldots j_p} a_{j_1j_2 \ldots j_p} x_{j_1} x_{j_2} \ldots x_{j_p} \]

Let’s take an arbitrary upward series of p indexes \( \underline{j}, \underline{j}, \ldots, \underline{j} \) from the set \( J_m \). Now, we will add all the summands within the sum III whose indexes \( j_1', j_2', \ldots, j_p' \) are permutations of \( j_1, j_2, \ldots, j_p \). The amount of those summands is \( p! \), where they make up a sum that can be presented as

\[ \sum_{\sigma \in S_p} \varepsilon_\sigma x_{s_1} x_{s_2} \ldots x_{s_p} \varphi (e_{j_1} e_{j_2} \ldots e_{j_p}) = \varphi (e_{j_1} e_{j_2} \ldots e_{j_p}) \varepsilon_{(j_1j_2 \ldots j_p)} \delta_{j_1j_2 \ldots j_p}, 1 \leq k \leq p, 1 \leq k \leq p. \]

By adding according to all growing indexes \( j_1, j_2, \ldots, j_p \) from the set \( J_m \), we obtain the sum I and equation II.

Now, we shall present the mapping of \( \varphi : E^p \rightarrow R \) in the form I with arbitrary coefficients \( a_{j_1j_2 \ldots j_p} \).

Let’s show that \( \varphi \) is anti-symmetric and p-linear.

Actually, \( \varphi \) is the sum of functions out of which each is in proportion with some determinant and that determinant is polilinear anti-symmetrization and, therefore, it represents an anti-symmetric form.

At the end, we need to mention that form I is unique, i.e. the equation II is certainly valid. Actually, if we put \( x_i = \ell_k, \quad k = 1, 2, \ldots, p \), then we get

\[ \Delta_{j_1j_2 \ldots j_p} \ell_{k_1} \ell_{k_2} \ldots \ell_{k_p} = \delta_{j_1k_1} \delta_{j_2k_2} \ldots \delta_{j_pk_p}. \]

Where \( \delta_{jk} \) are Kroneker symbols. From this it follows

\[ \varphi (x_1, x_2, \ldots, x_p) = a_{k_1k_2 \ldots k_p}, \]

i.e. coefficients II are found in one meaning form. This results in anti-symmetric p-forms \( \Delta_{j_1j_2 \ldots j_p} \) form a base in space \( A_p (R^m, R) \) and that dimension of that space is \( C_p^m \). This proves the theorem.

Let’s say that on \( R^m, p \) linear forms are given

\[ \varphi_i : R^m \rightarrow R, \quad i = 1, 2, \ldots, p. \]

From these linear forms we can make a p-linear form

\[ (x_1, x_2, \ldots, x_p) \rightarrow \varphi_1 (x_1) \varphi_2 (x_2) \ldots \varphi_p (x_p) \]

(2)

Anti-symmetrization of the form (2)

\[ (x_1, x_2, \ldots, x_p) \rightarrow \sigma \varepsilon_{\sigma} \varphi_1 (x_{\sigma_1}) \varphi_2 (x_{\sigma_2}) \ldots \varphi_p (x_{\sigma_p}) \]

We call EXTERNAL PRODUCT of the form (1) and we label it in the following way \( \varphi_1 \Lambda \varphi_2 \Lambda \ldots \Lambda \varphi_p \).

Accordingly, for any vector system \( x_1, x_2, \ldots, x_p \) from \( R^m \) the following equation is valid

\[ (\varphi_1 \Lambda \varphi_2 \Lambda \ldots \Lambda \varphi_p) (x_1, x_2, \ldots, x_p) = \varepsilon_{\sigma} \varphi_1 (x_{\sigma_1}) \varphi_2 (x_{\sigma_2}) \ldots \varphi_p (x_{\sigma_p}) \]

(4)

For two linear forms \( x \rightarrow \varphi (x), \quad y \rightarrow \Psi (y) \), \( x \in R^m, \quad y \in R^m \), the following equation is valid

\[ (\varphi \Lambda \Psi) (x,y) = \varphi (x) \Psi (y) - \varphi (y) \Psi (x) \]

Let’s say that \( t_1, t_2, \ldots, t_m \) is a base of a space \( R^m \). We will mark with \( (\xi_1) \Lambda (\xi_2) \Lambda \ldots \Lambda (\xi_m) \) represents a p-linear anti-symmetric form, which according to (3) represents anti-symmetrization of p-form \( (\xi_1) \Lambda (\xi_2) \Lambda \ldots \Lambda (\xi_m) \) represents a p-linear anti-symmetric form, which according to (3) represents anti-symmetrization of p-form \( (\xi_1) \Lambda (\xi_2) \Lambda \ldots \Lambda (\xi_m) \) represents a p-linear anti-symmetric form, which according to (3) represents anti-symmetrization of p-form

\[ \varphi = \sum a_{j_1j_2 \ldots j_p} \Lambda (\xi_1) \Lambda (\xi_2) \Lambda \ldots \Lambda (\xi_m), \quad 1 \leq j_1 < j_2 < \ldots < j_p \leq m \]

(5)

III. DIFFERENTIAL FORMS

Let’s say that \( D \) is an open set of the space \( R^m \).
Definition: By differential form of degree $p$ (or differential $p$-form) defined on $D$ and whose values are in $R$, we imply the following function:

$$\omega: D \rightarrow \mathcal{A}_p(D, R).$$

Function $\omega$ maps each point $x \in D$ in anti-symmetric $p$-form. Differential $p$-form is $n$ times differentiable if the function $\omega$ is $n$ times differentiable, where $n$ is a positive number or $+\infty$.

Set of all $n$ times differentiable $p$-forms on $D$ with values in $R$ we will mark with the symbol $\Omega^n_p D, R$. Set $\Omega^n_p D, R$ is vector space over the field $R$.

If $\omega \in \Omega^n_p D, R$, then with $\omega(x) \in \mathcal{A}_p(D, R)$, we can mark the values of the function $\omega(x) \in \mathcal{A}_p(R^n, R)$ on vector system $X_1, X_2, \ldots, X_p \in R^n$. Sometimes, those values are written with $\omega(x; X_1, X_2, \ldots, X_p)$.

**Theorem 3:** If $(\xi)$ is a linear form that joins each vector $X \in R^m$, its $i$ coordinate, then each anti-symmetric $p$-form, determined on $D$ with values in $R$, is presented as follows:

$$\omega(x) = \sum \omega_{j_1j_2 \ldots j_p} \xi_1 \Lambda \xi_2 \Lambda \ldots \Lambda \xi_p,$$

where $\omega_{j_1j_2 \ldots j_p} \in \mathcal{A}_p(D, \mathbb{R})$ has a dimension $C^n_p$ which means the equation (6). Vector space $\mathcal{A}_p(D, \mathbb{R})$ has the following form:

$$\omega(x) = \sum \omega_{j_1j_2 \ldots j_p} (x) \det (X_{j_1j_2 \ldots j_p}).$$

Right side of this equation is called canonical record of differential form. Its values on vector system $X_1, X_2, \ldots, X_p \in R^n$ are determined according to the formula:

$$\omega(x) = \sum \omega_{j_1j_2 \ldots j_p} \det (X_{j_1j_2 \ldots j_p}).$$

Let’s say that $\alpha \in \Omega^n_p D, R$ and $\beta(x)$ belongs to the space $\mathcal{A}_q(R^n, R)$. Then, its external product is:

$$\alpha \wedge \beta(x) \in \mathcal{A}_{p+q}(R^n, R).$$

By EXTERNAL PRODUCT of differential forms $\alpha$ and $\beta$ we imply differential form $(\alpha \wedge \beta)(x) \in \Omega^n_{p+q} D, R$ where the mapping $x \rightarrow (\alpha \wedge \beta)(x)$ on vector system $X_1, X_2, \ldots, X_{p+q}$ from $R^n$ is determined by:

$$(\alpha \wedge \beta)(x; X_1, X_2, \ldots, X_{p+q}) = \sum_{\sigma \in \mathfrak{S}_{p+q}} \epsilon_\sigma \alpha(x; X_{\sigma_1}, \ldots, X_{\sigma_p}) \beta(x; X_{\sigma_{p+1}}, \ldots, X_{\sigma_{p+q}}).$$

Where the summing is performed according to permutations of $\sigma$ set $\mathfrak{S}_{p+q}$ which meet the following condition $\sigma_1 < \sigma_2 < \ldots < \sigma_p$ and $\sigma_{p+1} < \sigma_{p+2} < \ldots < \sigma_{p+q}$.

Let’s say that $f : D \rightarrow R$, $D \in R^n$ is differentiable function.

In that case, its derivative $f' = \frac{\partial f}{\partial x_1} x_{dx_1} + \frac{\partial f}{\partial x_2} x_{dx_2} + \ldots + \frac{\partial f}{\partial x_m} x_{dx_m}$.

If there are $p$ scalar differential functions given $f_i : D \rightarrow R$, $D \in R^n$, $i=1,2,\ldots,p$, then observing differentials as differential forms of degree 1, their external product is represented as:

$$(d f_1 \wedge d f_2 \wedge \ldots \wedge d f_p)(x) = \sum \frac{D(f_1, f_2, \ldots, f_p)}{D(x_{f_1}, x_{f_2}, \ldots, x_{f_p})} d x_{f_1} \wedge d x_{f_2} \wedge \ldots \wedge d x_{f_p}.$$
form of the class $\mathcal{C}^0$ in some environment $U$ of $M$ multiplicity. It is required to determine the integral

$$\int_M \omega$$

Of the $p$-form $\omega$ on multiplicity $M$, where $(p)$ denotes the multiplicity of the integral.

Firstly, we shall observe one particular case, when the intersection of multiplicity $M$ with carrier of $p$-form is contained in a related open set $V \subset M$, for which the parametrization of the class $C^1$ is determined

$$\varphi : D \to V$$

where $D$ is related environment of null in $\mathbb{R}^n$.

Parametrization $t \to \varphi(t)$, $t \in D$, $t = (t_1, \ldots, t_p)$

we shall select in such a way that it is in accordance with given orientation $M$. It is obvious that $M \cap \text{supp} \omega$ – compact is contained in $V$, therefore, its original $\varphi^{-1}(M \cap \text{supp} \omega)$ – compact is contained in $D$. Let’s observe differential form $\varphi^\ast \omega$, defined on a set $D$.

It can be written as

$$f(t) \, dt_1 \Lambda \ldots \Lambda dt_p.$$ 

If $\varphi(t) = (\varphi_1(t), \ldots, \varphi_p(t))$, $t \in D$, then we complianly form the exchange of parameters

$$\varphi^\ast(\omega) = \sum \omega_{j_1 \ldots j_p} \varphi \frac{D(\varphi_{j_1} \ldots \varphi_{j_p})}{D t_1 \ldots t_p} \, dt_1 \Lambda \ldots \Lambda dt_p,$$

where $1 \leq j_1 < \ldots < j_p \leq n$.

And then

$$f(t) = \sum \omega_{j_1 \ldots j_p} \varphi \frac{D(\varphi_{j_1} \ldots \varphi_{j_p})}{D t_1 \ldots t_p} \, dt_1 \Lambda \ldots \Lambda dt_p.$$ 

Function $f$ is continuous on $D$ with compact carrier. Integral of the form $\omega$ on multiple $M$ is determined by the equation

$$\int_M \omega = \int_D f(t) \, dt_1 \Lambda \ldots \Lambda dt_p = \varphi \ast \omega,$$

Or by using the form (1) we get

$$\int_M \omega = \omega_{j_1 \ldots j_p} \varphi \frac{D(\varphi_{j_1} \ldots \varphi_{j_p})}{D t_1 \ldots t_p} \, dt_1 \Lambda \ldots \Lambda dt_p,$$

where $1 \leq j_1 < \ldots < j_p \leq n$.

In that case, it follows

$$f(t) \, dt_1 \Lambda \ldots \Lambda dt_p = f(t) \, dt_1 \ldots dt_n,$$

Where $\theta = +1$ if the base $\mathbb{R}^n$ is positive and $\theta = -1$ if the base $\mathbb{R}^n$ is positive. We need to prove that formula does not depend on the choice of parametrization $\varphi$.

**Theorem 4:** Let’s assume that the intersection $M \cap \text{supp} \omega$ of $p$-dimensional compact multiplicity $M \subset \mathbb{R}^n$ with the carrier of differential $p$-form $\omega$ of the class $\mathcal{C}^0$ is contained in related open set $V \subset M$. In that case, the equation (2) is valid for any parametrization: $D \to V$ of the class $C^1$.

**Proof:** Let’s assume that second parametrization $\psi : D' \to V'$ of the open set $V' \subset M$ that contains compact $M \cap \text{supp} \omega$ is given. Let’s say that $V_1 = V \cap V'$, $D_1 = \psi^{-1}(V_1)$, $D_1 \subset D$, $D_1' = \psi^{-1}(V_1)$, $D_1' \subset D'$. If $M \cap \text{supp} \omega$ is contained in the set $V$ and in $V'$, then $(M \cap \text{supp} \omega) \subset D_1$. Therefore, carrier of the form $\psi^\ast \omega$ is contained in $D_1$ and thus $(p) \, \varphi \ast \omega = (p) \, \psi \ast \omega$. (4)

For that reason,

$$\int_M \omega = \int_{D_1} \psi \ast \omega = \int_{D_1'} \psi \ast \omega.$$

If $\lambda : D_1' \to D_1$ some $C^\ast$ is differentiated and it retains the orientation, then the following equation $\psi = \varphi \lambda$ is correct on the set $D_1'$.

From this, it follows that $\psi^\ast \omega = \lambda^\ast (\varphi^\ast \omega)$. In accordance to the theorem about the exchange of variables in $p$ integral, we get

$$\int_{D_1} \psi \ast \omega = \int_{D_1'} \varphi \ast \omega.$$ 

Which proves the correctness of the definition of integral $\int_M \omega$ through the equation (2).

**REFERENCES**


