THE INFLUENCE OF THE VIBRATIONS UPON THE STRESS AND DEFORMATION STATES IN CASE OF THE LINEAR-ELASTIC CONNECTING ROD FOR A SLIDER CRANK MECHANISM R(RRT)

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Abstract. First of all, the movement equations of the linear-elastic straight kinematical elements in plan-parallel motion are presented. These are obtained by using the Hamilton's variation principle. In order to obtain the dynamic response, the Laplace integral transforms and the finite Fourier transforms are applied. Finally, the components of unit deformation tensor and those of the stress tensor, their variations depending on time and points distribution, in a concrete case, are given.

1. THE MOVEMENT MATHEMATICAL MODEL OF A KINEMATICAL ELEMENT LINEAR- ELASTIC SRAIGHT BAR TYPE.

By using the Hamilton's variation principle, the movement equations for kinematical elements linear-elastic straight bar type were inferred under the form of[1]:

$$[L]\{u\}+[M_4]\{a_0\}+\{V\}_1+[M_7]\{f\}+\{V_2\}=\{0\} \quad , \tag{1}$$

where $[L]\bullet$ is a differential operator defined by the expression:

$$[L]\bullet=[M_1]\frac{\partial^4\bullet}{\partial x^4}+[M_2]\frac{\partial^4\bullet}{\partial x^2\partial t^2}+[M_3]\frac{\partial^2\bullet}{\partial x^2}+[M_4]\frac{\partial^2\bullet}{\partial t^2}+[M_5]\frac{\partial\bullet}{\partial t}+[M_6]\bullet \tag{2}$$

In case of plan-parallel motion (fig.1), the matrices and the vectors from the relations (1) and (2) have the following form:

$$[M_1]=E\cdot I\cdot\begin{bmatrix}0&0\\0&1\end{bmatrix};[M_2]=\rho\cdot I\cdot\begin{bmatrix}0&0\\0&-1\end{bmatrix};[M_3]=E\cdot A\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix};[M_4]=\rho\cdot A\cdot\begin{bmatrix}-1&0\\0&1\end{bmatrix};[M_7]=\begin{bmatrix}1&0\\0&-1\end{bmatrix};$$

$$\{V_1\}=\rho\cdot A\cdot x\cdot\{\ \omega^2;\varepsilon\}^T;\{V_2\}=\left\{o;\frac{\partial m}{\partial x}\right\}^T;\{u\}=\{u_1;u_2\}^T;\{a_0\}=\{a_{01};a_{02}\}^T;\{f\}=\{f_1;f_2\}^T;\{0\}=\{0;0\}^T,$$

where: $\vec{u}$ - linear-elastic displacement;
ϖ - angular instantaneous velocity of the bar;
$\vec{\varepsilon}$ - angular instantaneous acceleration of the bar;
$\vec{a}_0$ - acceleration of the bar extremity O;
$\vec{f}(x,t)$ - exterior force in rapport a length unity;
$\vec{m}(x,t)$- exterior moment in rapport a length unity;
ρ - specific mass of the bar;
A - area of the transversal section of the bar;
E - the Young's modulus;
I_{zz} - the geometric inertia moment of the transversal section of the bar in relation to the neutral axis.

The matrices [M_5]and [M_6] have different form for the two variants, the coupled and the uncoupled variant, where the mathematical model (1) was inferred. For this reason, they are presented separately in table1.

Table 1

Matrix \ Variant	Coupled equations	Uncoupled equations
$[M_5]$	$2\rho \cdot A \cdot \omega \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$[0]$
$[M_6]$	$\rho \cdot A \cdot \begin{bmatrix} \omega^2 & \varepsilon \\ \varepsilon & -\omega^2 \end{bmatrix}$	$\rho \cdot A \cdot \omega^2 \cdot [M_7]$

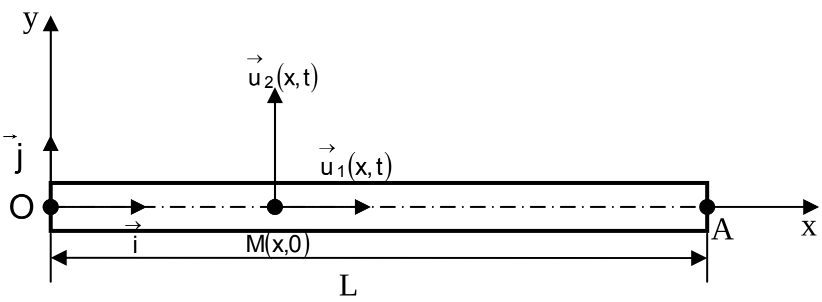

Fig.1 A bar with plan-parallel motion

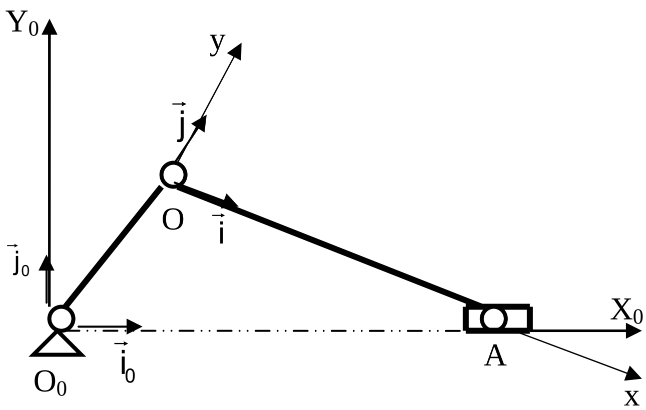

Fig. 2 R(RRT) mechanism

Grouping the connection terms between the longitudinal and the cross vibrations, as well those terms which confer to the mathematical model the quality of invariant in relation to time model it results:

$$[L_0]\{u\} + [M_4]\{a_0\} + \{V_1\} + [M_7]\{f\} + \{V_2\} + \{F\} = \{0\} \qquad (1')$$

where the differential operator $[L_0]\bullet$ is:

$$[L_0]\bullet = [M_1]\frac{\partial^4 \bullet}{\partial x^4} + [M_2]\frac{\partial^4 \bullet}{\partial x^2 \partial t^2} + [M_3]\frac{\partial^2 \bullet}{\partial x^2} + [M_4]\frac{\partial^2 \bullet}{\partial t^2}, \qquad (2')$$

and:

$$\{F\} = \{F_1, F_2\} = [L_1]\{u\} \quad , \qquad (3)$$

with:

$$[L_1]\bullet = [M_5]\frac{\partial \bullet}{\partial t} + [M_6]\bullet , \qquad (2")$$

contains the terms above mentioned.

Neglecting the vector {F} in (1'), it results:

$$[L_0]\{u\} + [M_4]\{a_0\} + \{V_1\} + [M_7]\{f\} + \{V_2\} = \{0\} \qquad (1")$$

which is an uncoupled model, linear and with constant coefficients in a first approximation. With $\{V_2\} = \{0\}$]i $\{f\} = \{0\}$, in (1'), it results the mathematical model of the free vibrations:

$$[L_0]\{u\} + [M_4]\{a_0\} + \{V_1\} + \{F\} = \{0\} \quad , \qquad (1"')$$

and with $\{F\} = \{0\}$, in (1"'), it results the mathematical model of the first approximation under the form:

$$[L_0]\{u\} + [M_4]\{a_0\} + \{V\}_1 = \{0\} \qquad (1^{IV})$$

2. THE ANALYTIC DETERMINATION OF THE DISPLACEMENTS FIELD.

By applying the unilateral Laplace transform in relation to time and the finite transforms in cosine and sine, to the first, respectively to the second equation, result the algebraic uncoupled systems having the displacements $u_{1,c}^{(1)}(n,s)$]i $u_{2,s}^{(1)}(n,s)$ in Laplace and Fourier images in cosine, respectively in sine, unknown. Then, inverting the Laplace and Fourier transforms, it results the solution in the first approximation $\{u^{(1)}(x,t)\}$. With $\{u^{(1)}(x,t)\}$ is calculated, in a first approximation, the vector $\{F^{(1)}\}$, which is introduced in the equation (1'). This way, it results the mathematical model in a second approximation. Solving this one, with the help of the integral transforms, is obtained the solution $\{u^{(2)}(x,t)\}$ in the second approximation. Resuming the repetitive process, it results the mathematical model in the "j" approximation:

$$[L_0]\{u^{(j)}(x,t)\} + [M_4]\{a_0\} + \{V\}_1 + [M_7]\{f\} + \{V_2\} + \{F^{(j-1)}\} = \{0\} \qquad (4)$$

where:

$$\{F^{(j-1)}\} = [L_1]\{u^{(j-1)}(x,t)\},\ j=1,2,...,n,\ \{F^{(0)}\} = \{0\} \qquad (5)$$

The solution in the "j" approximation should be:

$$u_1^{(j)}(x,t) = \frac{1}{L}\cdot u_{1,c}^{(j)}(o,t) + \frac{2}{L}\sum_{n=1}^{\infty} u_{1,c}^{(j)}(n,t)\cdot\cos(\alpha_n\cdot x) \qquad (6)$$

$$u_2^{(j)}(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} u_{2,s}^{(j)}(n,t)\cdot\sin(\alpha_n\cdot x), \qquad (7)$$

where $u_{1,c}^{(j)}(n,t)$ and $u_{2,s}^{(j)}(n,t)$ are the finite Fourier transforms in cosine, respectively in sine, of the elastic longitudinal, respectively cross displacements. The connecting rod OA being double jointed, the boundary conditions which have permitted the application of the two Fourier transform of the original functions and, respectively, for their Laplace images, were:

$$\frac{\partial u_1(0,t)}{\partial x} = \frac{\partial u_1(L,t)}{\partial x} = 0; u_2(0,t) = u_2(L,t) = 0; \frac{\partial^2 u_2(0,t)}{\partial x^2} = \frac{\partial^2 u_2(L,t)}{\partial x^2} = 0,$$

$$\frac{\partial u_1(0,s)}{\partial x} = \frac{\partial u_1(L,s)}{\partial x} = 0; u_2(0,s) = u_2(L,s) = 0; \frac{\partial^2 u_2(0,s)}{\partial x^2} = \frac{\partial^2 u_2(L,s)}{\partial x^2} = 0.$$

The boundary conditions were:

$$u_1(x,o) = 0; \frac{\partial u_1(x,0)}{\partial t} = 0; u_2(x,0) = 0; \frac{\partial u_2(x,0)}{\partial t} = 0$$

This way it results under the form of a primary approximation the transversal and longitudinal displacement fields in case of free vibrations of OA connecting rod of the R(RRT) mechanism from fig. 2.

3. STRESS AND STRAIN STATES

The components of the specific deformation tensor and of the stress tensor are given by:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), i, j = 1,2, \sigma_{i,j} = 2G\varepsilon_{ij} + \lambda\theta\delta_{i,j}, i, j = 1,2 \quad (8)$$

where:

$$\delta_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}; \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}; \theta = div\vec{u} = \sum_{i=1}^{2}\frac{\partial u_i}{\partial x_i}; u_1 = u_1^{(j)}(x,t); u_2 = u_2^{(j)}(x,t);$$

G- rigidity modulus;

ν - Poisson's coefficient of contraction;

Using (6) (7) and (8) result the components of the two tensors in "j" approximation.

$$\varepsilon_{11}^{(j)}(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}(-\alpha_n)\left[u_{1,c}^{(j)}(n,t)\right]\sin(\alpha_n x); \varepsilon_{22}^{(j)}(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\alpha_n u_{2,s}^{(j)}(n,t)\cdot\cos(\alpha_n \cdot x);$$

$$\varepsilon_{12}^{(j)}(x,t) = \frac{1}{L}\sum_{n=1}^{\infty}\alpha_n\left[u_{2,s}^{(j)}(n,t)\cos(\alpha_n x) - u_{1,c}^{(j)}(n,t)\sin(\alpha_n x)\right];$$

$$\sigma_{11}^{(j)}(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\alpha_n\left[\lambda\cdot u_{2,s}^{(j)}\cos(\alpha_n x) - (2G+\lambda)\cdot u_{1,c}^{(j)}(n,t)\sin(\alpha_n x)\right]; \quad (9)$$

$$\sigma_{22}^{(j)}(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\alpha_n\left[(2G+\lambda)\cdot u_{2,s}^{(j)}\cos(\alpha_n x) - \lambda\cdot u_{1,c}^{(j)}(n,t)\sin(\alpha_n x)\right];$$

$$\sigma_{12}^{(j)}(x,t) = \frac{2G}{L}\sum_{n=1}^{\infty}\alpha_n\left[u_{2,s}^{(j)}(n,t)\cos(\alpha_n x) - u_{1,c}^{(j)}(n,t)\sin(\alpha_n x)\right].$$

Under the form of the first approximation, the components of the two tensors are:

$$\varepsilon_{11}^{(1)}(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}(-\alpha_n)\left[a_{1,n} + a_{2,n}\cdot\cos^2(\omega_0 t) + a_{3,n}\cdot\cos(\omega_0 t) + a_{4,n}\cdot\cos(\omega_{n,1}t)\right]\cdot\sin(\alpha_n x);$$

$$\varepsilon_{22}^{(1)}(x,t) = \frac{2b}{L}\sum_{n=1}^{\infty}\frac{\alpha_n}{b_n}\left[\left(\sum_{j=1}^{4}b_{j,n}\right)\sin(\omega_0 t) + (b_{5,n}+b_{6,n})\sin(2\omega_0 t) + \left(\sum_{j=1}^{4}c_{j,n}\right)\sin(\omega_{n,2}t)\right]\cos(\alpha_n x);$$

$$\varepsilon_{12}^{(1)}(x,t) = \frac{1}{L}\sum_{n=1}^{\infty}\alpha_n\left\{\frac{b}{b_n}\left[\left(\sum_{j=1}^{4}b_{j,n}\right)\sin(\omega_0 t) + (b_{5,n}+b_{6,n})\sin(2\omega_0 t) + \left(\sum_{j=1}^{4}c_{j,n}\right)\sin(\omega_{n,2}t)\right]\cdot\right.$$
$$\left.\cdot\cos(\alpha_n x) - \left[a_{1,n} + a_{2,n}\cdot\cos^2(\omega_0 t) + a_{3,n}\cdot\cos(\omega_0 t) + a_{4,n}\cdot\cos(\omega_{n,1}t)\right]\cdot\sin(\alpha_n x)\right\};$$

$$\sigma_{11}^{(1)}(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\alpha_n\left\{\frac{\lambda b}{b_n}\left[\left(\sum_{j=1}^{4}b_{j,n}\right)\sin(\omega_0 t) + (b_{5,n}+b_{6,n})\sin(2\omega_0 t) + \left(\sum_{j=1}^{4}c_{j,n}\right)\sin(\omega_{n,2}t)\right]\cdot\right.$$
$$\left.\cdot\cos(\alpha_n x) - (2G+\lambda)\left[a_{1,n} + a_{2,n}\cdot\cos^2(\omega_0 t) + a_{3,n}\cdot\cos(\omega_0 t) + a_{4,n}\cdot\cos(\omega_{n,1}t)\right]\cdot\sin(\alpha_n x)\right\};$$

(9')

$$\sigma_{22}^{(1)}(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\alpha_n\left\{\frac{(2G+\lambda)b}{b_n}\left[\left(\sum_{j=1}^{4}b_{j,n}\right)\sin(\omega_0 t) + (b_{5,n}+b_{6,n})\sin(2\omega_0 t) + \left(\sum_{j=1}^{4}c_{j,n}\right)\sin(\omega_{n,2}t)\right]\cdot\right.$$
$$\left.\cdot\cos(\alpha_n x) - \lambda\left[a_{1,n} + a_{2,n}\cdot\cos^2(\omega_0 t) + a_{3,n}\cdot\cos(\omega_0 t) + a_{4,n}\cdot\cos(\omega_{n,1}t)\right]\cdot\sin(\alpha_n x)\right\};$$

$$\sigma_{12}^{(1)}(x,t)=\frac{2G}{L}\sum_{n=1}^{\infty}\alpha_n\left\{\frac{b}{b_n}\left[\left(\sum_{j=1}^{4}b_{j,n}\right)\sin(\omega_0 t)+(b_{5,n}+b_{6,n})\sin(2\omega_0 t)+\left(\sum_{j=1}^{4}c_{j,n}\right)\sin(\omega_{n,2}t)\right].\right.$$
$$\left.\cdot\cos(\alpha_n x)-\left[a_{1,n}+a_{2,n}\cdot\cos^2(\omega_0 t)+a_{3,n}\cdot\cos(\omega_0 t)+a_{4,n}\cdot\cos(\omega_{n,1}t)\right]\cdot\sin(\alpha_n x)\right\}.$$

where:

$$a_1=\frac{r(3r+4L)}{4};a_2=-\frac{3r^2}{4};a_3=-Lr;b=-Ar\rho\omega_0^2;\omega_{n,1}=\alpha_n\sqrt{\frac{E}{\rho}};\omega_{n,2}=\alpha_n^2\sqrt{\frac{EI}{\rho A}};\alpha_n=\frac{n\pi}{L};$$

$$a_{1,n}=\frac{\rho^2\omega_0^2r^2(\omega_0^2-\omega_{n,1}^2)}{\omega_{n,1}^2L^2\alpha_n^2(\rho\omega_0^2-E\alpha_n^2)(4\rho\omega_0^2-E\alpha_n^2)}\cdot\left\{L\alpha_n(\omega_{n,1}^2-4\omega_0^2)\sin(L\alpha_n)+2\omega_0^2[1-\cos(L\alpha_n)]\right\};$$

$$a_{2,n}=-\frac{\rho^2\omega_0^2r^2(\omega_0^2-\omega_{n,1}^2)}{L^2\alpha_n^2(\rho\omega_0^2-E\alpha_n^2)(4\rho\omega_0^2-E\alpha_n^2)}\cdot[2L\alpha_n\sin(L\alpha_n)+\cos(L\alpha_n)-1];$$

$$a_{3,n}=-\frac{\rho\omega_0^2r}{\alpha_n(\rho\omega_0^2-E\alpha_n^2)}\cdot\sin(L\alpha_n);$$

$$a_{4,n}=-a_{3,n}+\frac{\rho^2\omega_0^2r^2(\omega_0^2-\omega_{n,1}^2)}{\omega_{n,1}^2L^2\alpha_n^2(\rho\omega_0^2-E\alpha_n^2)(4\rho\omega_0^2-E\alpha_n^2)}\cdot$$
$$\cdot\left\{\omega_{n,1}^2L\alpha_n\sin(L\alpha_n)+(2\omega_0^2-\omega_{n,1}^2)[1-\cos(L\alpha_n)]\right\}$$

$$b_n=L\sqrt{EI}\cdot\alpha_n^4(E^2I^2\alpha_n^8-5AEI\rho\alpha_n^4\omega_0^2+4A^2\rho^2\omega_0^4);b_{1,n}=\sqrt{E^3I^3}\alpha_n^6\sin(L\alpha_n);b_{2,n}=-\sqrt{E^3I^3}\alpha_n^7;$$

$$b_{3,n}=-4A\rho\sqrt{EI}\cdot\alpha_n^2\omega_0^2\sin(L\alpha_n);b_{4,n}=4LA\rho\sqrt{EI}\cdot\alpha_n^3\omega_0^2;b_{5,n}=-\frac{1}{2}\sqrt{E^3I^3}\alpha_n^7r[1-\cos(L\alpha_n)];$$

$$b_{6,n}=\frac{1}{2}A\rho r\sqrt{EI}\cdot\alpha_n^3\omega_0^2[1-\cos(L\alpha_n)];c_{1,n}=EI\sqrt{A\rho}\cdot\alpha_n^4\omega_0\sin(L\alpha_n);$$

$$c_{2,n}=EI\sqrt{A\rho}\cdot\alpha_n^5\omega_0[L+r-r\cos(L\alpha_n)];c_{3,n}=4\sqrt{A^3\rho^3}\omega_0^3\sin(L\alpha_n);$$

$$c_{4,n}=-\sqrt{A^3\rho^3}\alpha_n\omega_0^3[4L+r-r\cos(L\alpha_n)].$$

4. GRAPHICAL REPRESENTATION

One can consider a concrete case for:
L=0,445[m];r=0,04[m]; ω_0 =76,61[s^{-1}];b=0,018[m];h=0,005[m]; E=$2,1\cdot10^{11}$ [N/m^2];
ρ=7800[Kg/m^3]; G=$8,1\cdot10^{10}$ [N/m^2]; ν=0,3.

Using (9’) and the above data one can obtain the graphical representations of the two tensors components:

$$\varepsilon_{ij}=\varepsilon_{ij}^{(1)}(x,t),\ \sigma_{ij}=\sigma_{ij}^{(1)}(x,t),\ \varepsilon_{ij}=\varepsilon_{ij}^{(1)}\left(\frac{L}{3},t\right),\ \sigma_{ij}=\sigma_{ij}^{(1)}\left(\frac{L}{3},t\right),\ i,j=1,2\ .$$

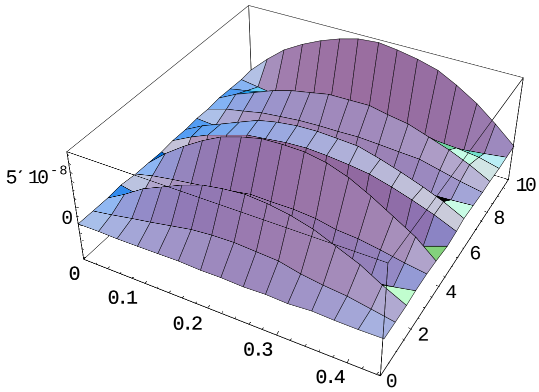

Fig. 3 $\varepsilon_{11} = \varepsilon_{11}^{(1)}(x,t)$

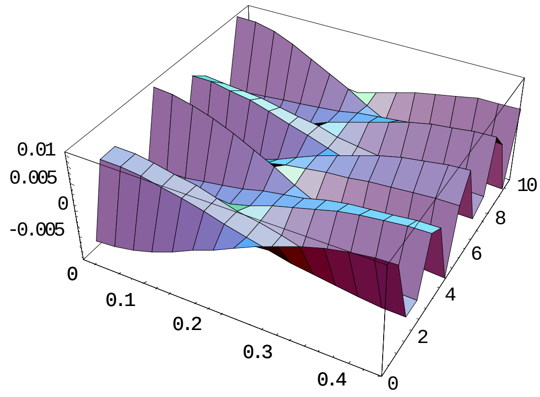

Fig.4 $\varepsilon_{22} = \varepsilon_{22}^{(1)}(x,t)$

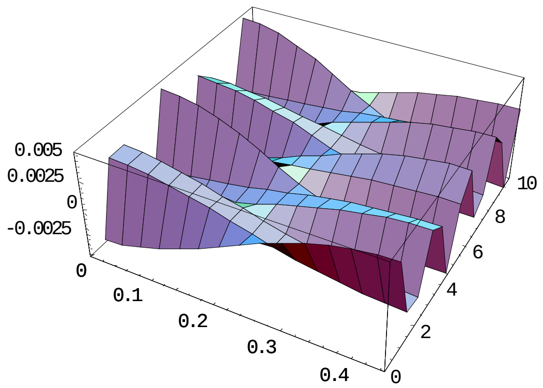

Fig. 5 $\varepsilon_{12} = \varepsilon_{12}^{(1)}(x,t)$

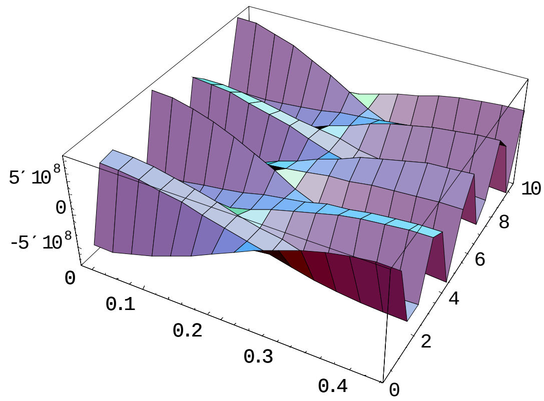

Fig. 6 $\sigma_{12} = \sigma_{12}^{(1)}(x,t)$

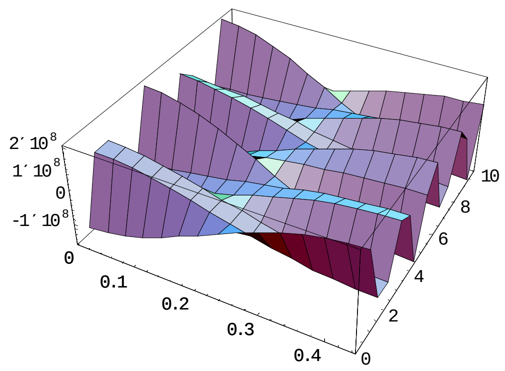

Fig. 7 $\sigma_{11} = \sigma_{11}^{(1)}(x,t)$

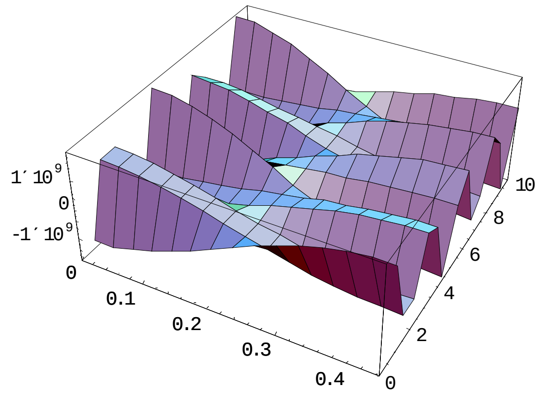

Fig. 8 $\sigma_{22} = \sigma_{22}^{(1)}(x,t)$

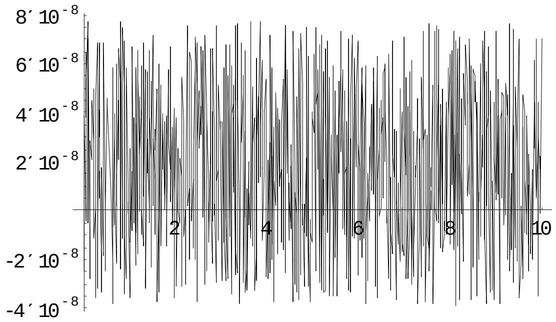

Fig. 9 $\varepsilon_{11} = \varepsilon_{11}^{(1)}\left(\frac{L}{3}, t\right)$

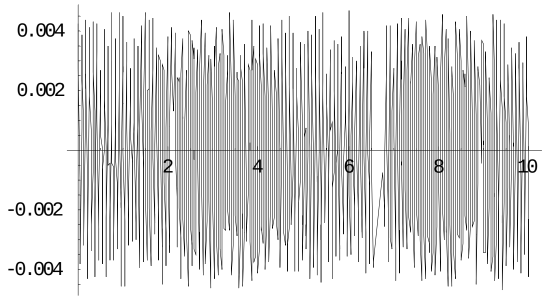

Fig.10 $\varepsilon_{22} = \varepsilon_{22}^{(1)}\left(\frac{L}{3}, t\right)$

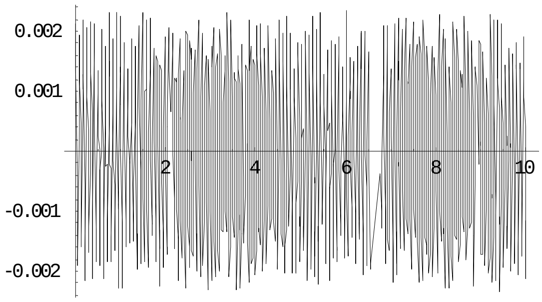

Fig.11 $\varepsilon_{12} = \varepsilon_{12}^{(1)}\left(\frac{L}{3}, t\right)$

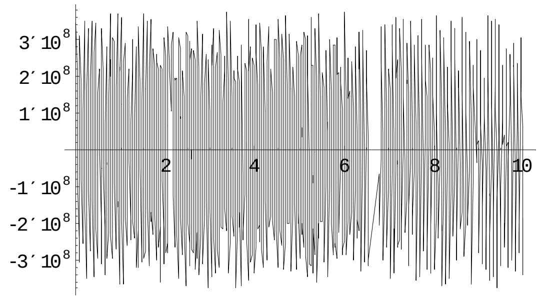

Fig.12 $\sigma_{12} = \sigma_{12}^{(1)}\left(\frac{L}{3}, t\right)$

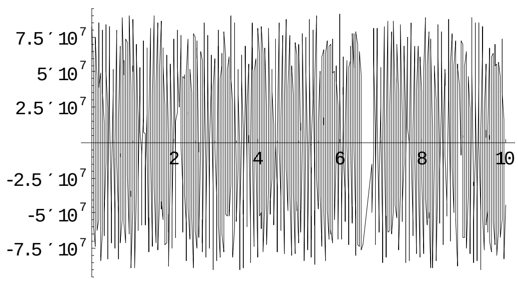

Fig.13 $\sigma_{11} = \sigma_{11}^{(1)}\left(\frac{L}{3}, t\right)$

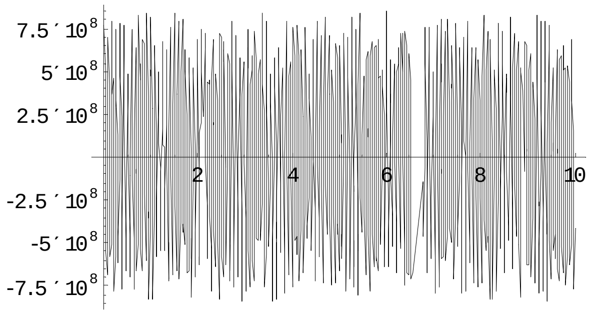

Fig.14 $\sigma_{22} = \sigma_{22}^{(1)}\left(\frac{L}{3}, t\right)$

5. CONCLUSIONS

The relations (9') indicate the important influence of the kinematical parameters over the displacements fields of the vibrating connecting rod and thus over the components of the specific deformations tensor and the stress tensor. Therefore, while designing, for the dimensioning calculus it is necessary to use relations similar to the (9') expressions from this paper.

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