ANNALS of the ORADEA UNIVERSITY.
Fascicle of Management and Technological Engineering

PSEUDOSPECTRA – APPLICATIONS TO ROBUST STABILITY AND CONTROL OF DYNAMICAL SYSTEMS

Dumitru NICOARA
Prof. univ. dr. mat., Transilvania University of Brasov
tnicoara@unitbv.ro

Abstract. In this paper we briefly sketch the connections of pseudospectra with related quantities, in particular, the distance to instability, the **H**$_\infty$ norm, and distance to uncontrollability. Pseudospectra and related quantities for non-normal matrices and operators where first investigated in the 1970s and 1980s and became a standard tool in the 1990s, with applications in mechanics, numerical analysis, operator theory, control theory, differential and integral equation. In all of this fields it has been found that in case of pronounced non-normality, eigenvalues and eigenvectors alone do not always reveal much about the aspects of the behavior of a matrix or operator that matter in applications, including phenomena of stability, convergence, and resonance, and that pseudo-eigenvalues and pseudo-eigenvector may do better.

1. INTRODUCTION

The pseudospectrum (set of pseudo-eigenvalues) is a powerful modeling tool: for example, large real parts of pseudo-eigenvalues (rather than eigenvalues themselves) often reveal the behavior of dynamical systems governed by A, $\dot{x} = Ax$. Trajectories of this system all converge to the origin exponentially if and only if the *spectral abscissa* $\alpha(A) = \max \operatorname{Re} \Lambda_0(A)$ is strictly negative. But as we observed, this condition is not robust: even if it holds, the *pseudospectral absciss* $\alpha_\varepsilon(A) = \max \operatorname{Re} \Lambda_\varepsilon(A)$ may be positive for small $\varepsilon > 0$. In other words, nearby matrices may not be stable, and related trajectories of $\dot{x} = Ax$ may have large transient peaks. This argument suggests the pseudospectral abscissa $\alpha_\varepsilon(A)$ (for some small $\varepsilon > 0$) is a better measure of system decay than the spectral abscissa.

Many classical problems of robustness and stability in control theory aim to move the eigenvalues of a parameterized matrix into some prescribed domain of the complex plane. However, a simple consideration of the spectrum alone has serious drawbacks in many contexts. As Trefethen and others have pointed out (see [10] and the references therein), pseudospectra of a matrix (sets of eigenvalues of all matrices within certain distances) are more informative and more robust in many modeling frameworks, and in particular as indicators of stability, of robustness [2], [3] and of controllability [8].

2. NOTATION

We consider a matrix *A* in the space of $n \times n$ complex matrices **M**^{n}**.** We denote the spectrum of *A* by $\Lambda = \Lambda(A)$,

$$\Lambda(A)=\{\lambda\in C:\det(A-\lambda I)=0\}=\{point\ where\ (A-\lambda I)^{-1}\ is\ undefinined\} \quad (1)$$

and we denote by $\alpha=\alpha(A)$ the *spectral abscissa* of *A,* which is the largest of the real parts of the eigenvalues.

$$\alpha_{\varepsilon}=\sup\{\operatorname{Re} z : z\in\Lambda(A)\} \quad (2)$$

For a real ε > 0, the ε-*pseudospectrum* of *A* is the set

$$\Lambda_{\varepsilon}=\{z\in C : z\in\Lambda(X) \text{ where} \|X-A\|<\varepsilon\} \quad (3)$$

Throughout, $\|\cdot\|$ denotes the operator 2-norm on $\mathbf{M}^n$. Any element of the pseudospectrum is called a *pseudo-eingenvalue.*

The ε-pseudospectral abscissa α_{ε} is the maximum value of the real part over the pseudo-spectrum:

$$\alpha_{\varepsilon}=\sup\{\operatorname{Re} z : z\in\Lambda_{\varepsilon}\}. \quad (4)$$

We call this optimization problem *the pseudospectral abscissa problem.* Note $\Lambda_0=\Lambda;\alpha_0=\alpha$.

If $\sigma_{\min}(A)$ denotes the smallest singular value of *A* then we have a useful caracterization of the pseudospectrum

$$\Lambda_{\varepsilon}(A)=\{z\in C : \sigma_{\min}(zI-A)\le\varepsilon\} \quad (5)$$

Thus the pseudospectra of *A* are the sets in the *z*-plane bounded by level curves of the function $g(z)=\sigma_{\min}(zI-A)=\|(A-zI)^{-1}\|^{-1}$, where we interpret the right-hand side as zero when $z\in\Lambda(A)$.

We need some more notations. First, consider the *Lyapunov condition factor* for a matrix $A\in\mathbf{M}^n$, defined by

$$L(A)=\inf\{\gamma>1 : \gamma I\succ H\succ\gamma^{-1}I,\ H\succ A^{*}H\,A,\ H\in\mathbf{H}^n\} \quad (6).$$

We compare this condition factor with the *power bound*

$$P(A)=\sup\{\|A^k\| : k=1,2,...\} \quad (7)$$

as well as with a third quantity, the *Kreisse constant*, defined in terms of pseudospectra,

$$K(A)=\sup\{\frac{|z|-1}{\varepsilon} : \varepsilon>0,\ z\in\Lambda_{\varepsilon}(A)\} \quad (8)$$

which is a bound on the linear rate at which the ε-pseudospectrum bulges outside the unit disk, as a function of ε.

3. PSEUDOSPECTRA AND RELATED IDEAS ON STABILITY

The pseudospectral abscissa is related to several other functions important for stability analysis. In this section we briefly sketch the connections with the *distance to instability*.

A matrix *A* is stable if the spectral abscissa of *A* satisfies $\alpha(A)<0$, in other words, all the eigenvalues have strictly negative real part. If *A* is stable, its distance to the set of unstable matrices [11], also known as the *complex stability radius* [5] is

$$\beta(A)=\min\{\|X-A\| : X\in\mathbf{M}^n,\ \alpha_{\varepsilon}(X)>0\}=\min_{z\in C}\{\sigma_{\min}(A-zI) : \operatorname{Re} z\ge 0\}. \quad (9)$$

It is now easy to check the relationship

$$\beta(A) \le \varepsilon \Leftrightarrow \alpha_{\varepsilon}(A) \ge 0, \qquad (10)$$

and more generally, for any real x

$$\beta(A - xI) \le \varepsilon \Leftrightarrow \alpha_{\varepsilon}(A) \ge x.$$

For a matrix A that is not stable, we have $\beta(A) = 0$, while for stable A, $\alpha_{\beta(A)}(A) = 0$, and the minimization (9) may as well be done over z on the imaginary axis. A matrix $A \in \mathbf{M}^n$ has an eingenvalue outside the closed unit disk exactly when the iteration $x_{k+1} = Ax_k$ exhibits exponential growth. These equivalent proprieties can be checked, for discrete –time systems, using the following well known result:

Theorem 1 (Lyapunov, 1893) For any matrix $A \in \mathbf{M}^n$, the following properties are equivalent:

1^0. All eigenvalue of A lie in the open unit disk;

2^0. $\|A^k\| \to 0$ exponentially as $k \to \infty$;

3^0. There exist a matrix $H \succ 0$ in $\mathbf{H}^n$ such that $H \succ A^* HA$.

For two matrices A and B in the space $\mathbf{H}^n$ of n-by-n Hermitian matrices, we write $A \succ B$ to mean $A - B$ is positive definite.

This result is deceptive for two reasons. The first difficulty is on of *robustness*. Because the dependence of the eigenvalues on A is not Lipchitz [8], when a matrix A is close to a matrix with multiple eigenvalue, unexpectedly small perturbations to A can destroy the stability. Secondly, there is the difficulty of *transient growth*: even if $\|A^k\| \to 0$ asymptotically, intermediate values of $\|A^k\|$ may be very large [9].

These difficulties can be quantified by the following classical result in numerical analysis [7].

Theorem 2 (Kreiss, 1962). Over any set of matrices in $\mathbf{M}^n$, the Kreiss constant k, the power bound P, and the Lyapunov condition factor L are either all uniformly bounded above or all unbounded.

Results parallel to Theorems 1 and 2 hold for the continuous-time version. In that case we are interested in bounding $\|e^{tA}\|$ $(t \ge 0)$, the left half plane takes the place of the unit disk, and the Lyapunov inequality becomes $H \succ 0$, $0 \succ A^* H + HA$. The relevant pseudospectral quantity is $\varepsilon^{-1} \operatorname{Re} z,\ (z \in \Lambda_{\varepsilon}(A))$.

The relationships between the quantities $\alpha_{\varepsilon}(A)$ and $\beta(A)$ and the *robustness* of the stability of A with respect to perturbation are clear from the definitions. Less obvious is that these functions also measure the *transient response* of the associated dynamical system $\dot{x} = Ax$ [31]. By choosing ε < $\beta(A)$ in $\alpha_{\varepsilon}(A)$,we place more emphasis on asymptotic behavior than when we choose $\varepsilon = \beta(A)$.Stated differently, we choose ε according to the size of perturbation we are prepared to tolerate, and measure what kind of asymptotic response this allows us to guarantee, instead of measuring the largest perturbation that can be tolerated while still guaranteeing stability. Efficient algorithms to compute the pseudospectral abscissa $\alpha_{\varepsilon}(A)$ and the complex stability radius $\beta(A)$ are available. Fast algorithms based on computing eigenvalues of Hamiltonian matrices is available in the MATLAB control toolboxes.

4. PSEUDOSPECTRA AND RELATED IDEAS ON CONTROL THEORY

The analogous question for controllability asks for *distance to uncontrollability*, the distance from the pair (*A, B*) to the nearest pair ($A^{/}$, $B^{/}$) corresponding to an uncontrollable systems.

For real matrices *A* and *B* of size $n \times n$ and $n \times m$ respectively, consider the control system defined by

$$\dot{x} = Ax + Bu \tag{11}$$

This system is said be state controllable at $t = t_0$ if there exist a piecewise continuous input ***u***(*t*) that will drive the initial state ***x***(t_0) to any final state ***x***(t_f). Classical theory provides a simple characterization of controllability. The above system is controllable exactly when the matrix [*A*−*zI* *B*] has full row rank, *n*, for all scalars $z \in$ C.

A small distance to uncontrollability correlates with various difficulties for the control system, including numerical challenges for associated “pole placement” problems. A simple argument [4] shows that the distance to uncontrollability is given by

$$\min_{z \in \mathbf{C}} \sigma_{\min}[A - zI \,|\, B]. \tag{12}$$

The function to be minimized in the expression (12) has lower level sets of the form

$$\{z \in \mathbf{C} : \sigma_{\min}[A - zI \mid B] \le \varepsilon\} \tag{13}$$

which is the pseudospectra, when matrix *B* is empty. Substantial insight is gained from examples, so consider the parameterized matrix pair, [4],

$$(A, B) = \left[\begin{bmatrix} 1 & 1 & 2 & 3 \\ -1 & 1 & 4 & 5 \\ 0 & \eta_1 & 1 & 2 \\ \eta_2 & 0 & -2 & 1 \end{bmatrix} \middle| \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right], \tag{14}$$

where η_1 and η_2 are real parameters. Figures 1 and 2 show pseudospectra (produced using by T.Wright’s software EigTool [6]) for, respectively, the controllable pair (14) when $\eta_1 = \eta_2 = 1$ and the un controllable pair (14) when $\eta_1 = \eta_2 = 0$.

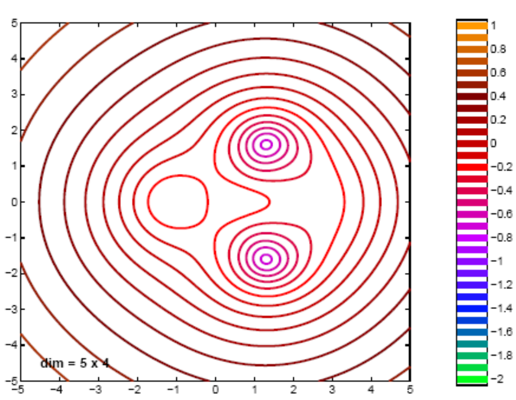

Figure 1: Pseudospectra for the controllable pair (14) with $\eta_1 = \eta_2 = 1$

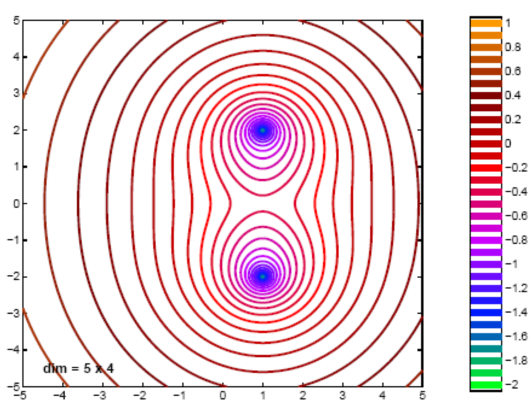

Figure 2: Pseudospectra for the uncontrollable pair (14) with $\eta_1 = \eta_2 = 0$

The horizontal and vertical axes in the figures show the real and imaginary parts of *z*. The legends on the right sides of the figures show the contour heights (values of ε) on a log 10 scale, with both plots using the same scale. In Figure 1, the "pseudospectral landscape" has three local minima and one can estimate that the global minimum value (by definition, the distance to uncontrollability) is about $10^{-0.75}$ (in fact, it is 0.1872). In Figure 2, there are only two local minima (forming a complex conjugate pair), and one can see that the contours drop to much lower values (in fact, it is easy to check that the minimal value of (1.1) is zero at the points $z = 1 \pm 2i$).

In Figure 1, the pseudospectra contain up to, but no more than, three connected components, depending on the choice of ε, while the pseudospectra in Figure 2 contain up to, but no more than, two connected components. Maximization of the distance to uncontrollability for smoothly varying parameterized pair (*A, B*)(*x*) over vector of free parameters $x \in \mathbf{R}^k$, is given with two algorithms namely the Trisection Variant of Gu's Algorithm and the BFGS/Gu Hybrid, [4]. Matlab implementations of the algorithms are freely available: http://www.cs.nyu.edu/faculty/overton/software.

The idea of the $\mathbf{H}_\infty$ norm of the transfer matrix is also closely related. Consider the linear tine-invariant dynamical system

$$\dot{x} = Ax + u \tag{15}$$

where *x* denotes the state vector (in this simple case coinciding with the output) and *u* denote the control (input) vector. The transfer matrix of this system is the function $H(s) = (sI - A)^{-1}$. Assuming the matrix *A* is stable, the corresponding $\mathbf{H}_\infty$ norm is defined by

$$\|H\|_\infty = \sup_{\omega \in \mathbf{R}} \sigma_{\max}(H(i\omega)) \tag{16}$$

where $\sigma_{\max}$ denote the largest singular value. Clearly

$$\|H\|_\infty = \sup_{\omega \in \mathbf{R}} \frac{1}{\sigma_{\min}(A - i\omega I)}$$

so $\|H\|_\infty < \varepsilon^{-1}$ if and only if we have

$$\sigma_{\min}(A - i\omega I) > \varepsilon \text{ for all } \omega \in \mathbf{R}.$$

In summary:

$$\alpha_\varepsilon(A) < 0 \Leftrightarrow \|H\|_\infty < \frac{1}{\varepsilon}. \tag{17}$$

An important topic in robust control has been the design of controllers that minimize the $\mathbf{H}_\infty$ norm [8]. In the language above, this corresponds to choosing the parameters defining the stable matrix *A* in order to maximize the minimum value of $\sigma_{\min}(A - zI)$ as *z* varies along the imaginary axis. We first fix the level of robustness ε, and then vary *A* to move the corresponding pseudospectrum as far as possible to the left in the complex plane. In other words, we try to maximize a real parameter *x* such that the $\mathbf{H}_\infty$ norm corresponding to the shifted matrix *A-xI* is never more than ε^{-1}.

A particular important example, in control theory, is stabilization by static output feedback: given a *n*-by-*n* matrix *A,* an *n*-by-*r* matrix *B* and *s*-by-*n* matrix *C*, find (if possible) an *r*-by-*s* matrix *K* such that *A+BKC* is stable.

Stabilization by static output feedback (SOF) is a long-standing open problem in control. Robust stabilization further demands stability in the presence of perturbation and satisfactory transient as well as asymptotic system response. In [8], are formulated two related nonsmooth, nonconvex optimization problems over K, respectively with the following objectives: minimization of the ε-pseudospectral abscissa of $A+BKC$, for fix $\varepsilon \geq 0$, and minimization of the complex stability radius of $A+BKC$:

$$\min_{K\in \mathbf{R}^{r\times s}} \alpha_{\varepsilon}(A+BKC) \quad \text{(a)} \qquad\qquad \min_{K\in \mathbf{R}^{r\times s}} -\beta(A+BKC) \quad \text{(b)} \qquad\qquad (18)$$

For modest-sized systems, local optimization can be carried out from large number of starting points with no difficulty.

In [4] is applied gradient sampling optimization to many static output feedback (SOF) stabilization problems published in control literature. We show results for a turbo-generator model [6], [4] that allows us to show different choices of the optimization objective lead to stabilization with qualitatively different properties, conveniently visualized by pseudospectral plot. . Figures 3 through 6 shows pseudospectral plot in the complex plane, showing, for particular K, the boundary of $\Lambda_{\varepsilon}(A+BKC)$ for four different values of ε. The legend at the right of each figure shows the logarithms (base 10) of the four values of used in the plot. A particular pseudospectrum $\Lambda_{\varepsilon}(BKC)$ may or may not be connected, but each connected component must contain one or more eigenvalues, shown as solid dots. The figures do not show all 10 eigenvalues; in particular, they do not show conjugate pair of eigenvalues with large imaginary part, whose corresponding pseudospectral components are well to the left of the ones appearing in the figures. In Figures 5 and 6, the smallest real eigenvalue is also outside the region shown.

The pseudospectral contours were plotted by T. Wright's EigTool, an extremely useful graphical interface for interpreting spectral and pseudospectral properties of nonsymmetric matrices [13]. Figure 3 shows the pseudospectra of the original matrix A, that is, with $K = 0$. Although is stable, since its eigenvalues are to the left of the imaginary axis, it is not robustly stable, since three connected components of the 10^{-2}-pseudospectrum cross the imaginary axis. Figure 4 shows the pseudospectra of $A+BKC$ when solves (18 a) with ε = 0, or, in other words, when the rightmost eigenvalue of $A+BKC$ is pushed as far as possible to the left. Notice that six eigenvalues of $A+BKC$ are now arranged on line parallel to the imaginary axis, and that two of the conjugate pairs are quite close to each other, indicating the possibility that at an exact minimizer there is double conjugate pair of eigenvalues as well as simple conjugate pair with thesamereal part. The10^{-2}-pseudospectrum is now contained in the left half-plane, but the $10^{-1.5}$-pseudospectrum is not.

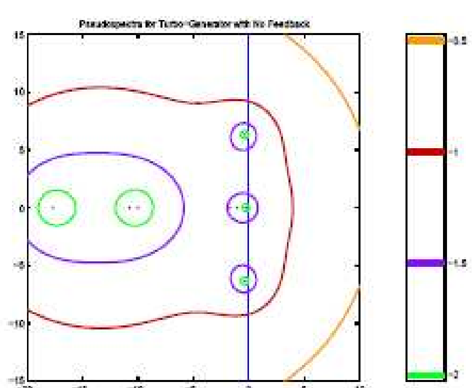

Fig. 3. Pseudospectra for Turbo-Generator with No Feedback

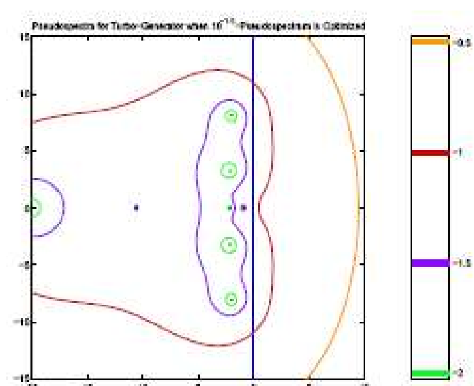

Fig. 5. Pseudospectra for Turbo- Generator when 10−1.5-Pseudospectrum is optimized

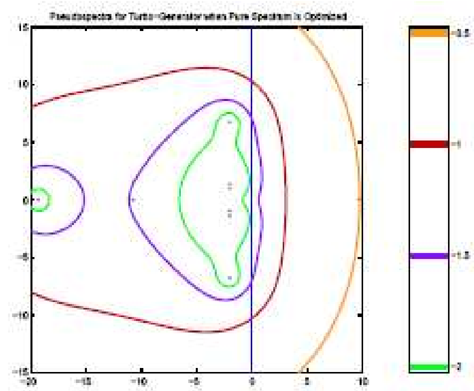

Fig. 4. Pseudospectra for Turbo-Generator when Pure Spectrum is Optimized

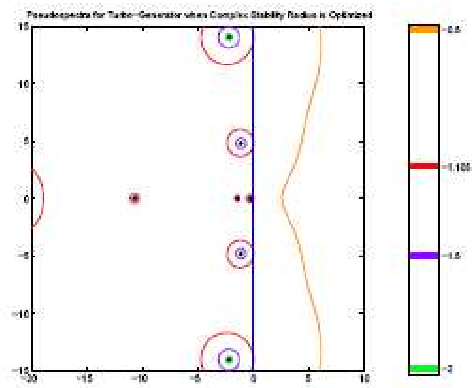

Fig. 6. Pseudospectra for Turbo-Generator when Complex Stability Radius is Optimized

Figure 5 shows the pseudospectra of $A + BKC$ when K solves (4) $\varepsilon = 10^{-1.5}$,or, in other words, when the rightmost part of the $10^{-1.5}$-pseudospectrum of $A + BKC$ is pushed as far as possible to the left. Now the $10^{-1.5}$-pseudospectrum is to the left of the imaginary axis, but the eigenvalues have moved back towards the right, compared to Figure 4. There is apparently a three-way tie for the maximizing z in (4) at the local minimizer — one real value in its own small pseudospectral component, and two conjugate pairs in a much larger pseudospectral component.

Figure 6 shows the pseudospectra of $A + BKC$ when K solves (18 b) (maximizes the complex stability radius β), or, in other words, when K is chosen to maximize the ε for which the ε-pseudospectrum of $A + BKC$ is contained in the left half-plane. For this optimal value, $\varepsilon = 10^{-1.105}$, the ε-pseudospectrum is tangent to the imaginary axis at five points, a real point and two conjugate pairs, indicating (as previously) a three-way tie for the minimizing z in (18 b). Each minimizing z has its own pseudospectral component. This ε-pseudospectrum crosses the imaginary axis in the previous three figures. On the other hand, the $10^{-1.5}$-pseudospectrum is now further to the right that it was in Figure 5.

5. CONCLUSIONS

Classical analysis of linear models is based upon eigenvalues, and for many problems across mathematics, science, and engineering, such analysis is successful. This is most notably true for self-adjoint matrices and operators, which possess a basis of orthogonal eigenvectors. Areas of successful applications of eigenvalue techniques include acoustics, structural analysis, quantum mechanics, low-Reynolds-number fluid mechanics, and numerical analysis.

In recent decades, recognition has grown that one must proceed with greater caution when a matrix or operator lacks an orthogonal basis of eigenvectors. Such operators are called non-normal, and this property can lead to a rich variety of behavior. For example, non-normality can be associated with transient behavior that differs entirely from the asymptotic behavior suggested by eigenvalues. Such transients may manifest themselves in slow convergence of iterative processes, in nearness to instability, and in the transition to turbulence in fluid flow.

Numerous tools have been suggested to describe non-normality and analyze its effects. These include classical tools of matrix and operator theory, such as the numerical range, the angles between invariant subspaces, and the condition numbers of eigenvalues. This paper is devoted to describing and illustrating pseudospectra and the connections of pseudospectra with related quantities, in particular, the distance to instability, the $\mathbf{H}_\infty$ norm, and distance to uncontrollability.

6. REFERENCES

[1] Blondel, V., Gevers, M., Linquist, A., "Survey on the state of systems and control", *European Journal of Control* **1**, 5-23, 1995.
[2] Burke,J.V., Lewis, A.S., Overton, M.L., Optimal stability and eigenvalue multiplicity", *Fundation of Computational Mathematics*, **1**:205-225,2001.
[3] Burke,J.V., Lewis, A.S., Overton, M.L., "Two methods for optimizing matrix stability", *Linear Algebra and its Applications*, **351-352**, 117-145.
[4] Burke,J.V., Lewis, A.S., Overton, M.L., "A nonsmoot, nonconvex optimization approach to robust stabilization by static output feedback", http://www.cs.nyu.edu/faculty/Overton
[5] Hinrichsen, D., and Pritchard, A. J., "Stability radii of linear systems", *Systems and Control Letters,* **7**:1-10, 1986.
[6] Hung, Y.S., MacFarlane, G.J., "Multivariabile feedback: A quasi-classical approach" Springer-Verlag, 1982.
[7] Kreiss, H.O., Uber die Stabilitäsdefinitions für Differenzengleichungen die partielle Diferentzialgleichungen approximieren, *BIT*, **2**:153-181, 1962.
[8] Lewis,A.S., Overton,M.L., "Eigenvalue optimization", *Acta Numerica,* **5**:149-190,1996.
[9] Trefethen, L.N., "Computation of Pseudospectra *Acta Numerica* , Cambridge University Press, 1-48,1999.
[10] Trefethen, L.N., "Pseudospectra of linear operators ", *SIAM Review* **39**, 383-406, 1997.
[11] Van Loan, C. F., " How near is a stable matrix to an unstable matrix?", *Comtemporary Mathematics,* 47:465-477, 1985.
[12] Toh, K.C., and Trefethen, L.N., "Computation of pseudospectra by the Arnoldi iteration" *SIAM J. Sci. Comput.* **17**, 1-15, 1996.
[13] Wright, T., EigTool, http://web.comlab.ox.ac.uk/projects/pseudospectra/