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A PURIFICATION THEOREM FOR EXTENDED FORM PERFECT INFORMATION GAMES

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Abstract. A theorem of purification related to the equilibrium property for the tube shaped perfect information game, is derived, in order to reduce the size of the algorithm of finding the saddle point for the decision function involved in this type of problem.

Key words. Extended form perfect information game, saddle – point of a function with respect to a set, tube shaped game, weakly dominated row.

1. INTRODUCTION AND BASIC CONCEPTS

Generally, each minimax theorem for a class of functions should be acompanied by a method of finding the saddle point. The purification theorems are meant to reduce the size of the solution and, generally, they accompany the saddle point theorems.

In this paper we intend to prove a purification theorem related to a minimax theorem, proved in [2], that applies to other cases than those satisfying the conditions, which Kalmar stated in [5] for chess. We shall refer to the class of perfect information games satisfying a condition, which we call "the tube condition", introduced in [2]. This type of games includes the Japanese go, but does not contain the chess. It also includes games arising in economic relations and in military operations as well. The extended form of perfect information games is used in order to avoid the loss of cases and information that may occur during the reduction process, as shown in [4]. The concept of saddle point with respect to a given set, introduced in [3], is crucial allover the proof of the purification theorem.

Now, let us recall the concept of saddle – point and equilibrium that we are going to use in what follows. For the beginning, let us consider two nonempty sets,

A and B and a nonempty subset $M \subseteq A \times B$. We denote $M_1 = pr_A M$ and $M_2 = pr_B M$. For a given point $(p, q) \in M$, we denote

$$M_p = \{y \in M_2 \mid (p, y) \in M\} \text{ and } M_q = \{x \in M_1 \mid (x, q) \in M\}.$$

Obviously, $M_p \subseteq B$ and $M_q \subseteq A$. Let us consider a function $f : A \times B \to \mathbf{R}$. The concept of saddle – point of a function with respect to a given set was first introduced in [3].

Definition 1.1. *A point* $(p, q) \in M$ *is said to be a saddle – point of the function* f *with respect to* M *if*

$$f(x, q) \le f(p, q) \le f(p, y),$$

for every $x \in M_q$ *and* $y \in M_p$.

It was proved in [3] that the following equality is characterising a saddle – point of a function f with respect to a set M:

$$\underline{v} = \max_{x \in A} \min_{y \in M_x} f(x, y) = \min_{y \in B} \max_{x \in M_y} f(x, y) = \overline{v}.$$

We use the description from [4] of the extensive form of a perfect information game. Let $\{a, b\}$ be a set of players. A perfect information game is represented by a tree given by a finite set of nodes N, which are partially ordered by $\prec$. The initial node of this tree is considered to be the least element of N with respect to this relation and is denoted by O. The set of terminal nodes are

$$Z = \{ z \in N \mid z \prec z' \Rightarrow z = z' \}$$

and $X = N \setminus Z$ is the set of non–terminal nodes. For every $x \in X$, the symbol $\{x\}$ is denoting the corresponding information set, sometimes identified with x itself. The set of the immediate successors of x, denoted by $C(x)$, is the set of choices available at node $x \in X$. The player that moves at each information set is designed by a mapping $m : X \to \{a, b\}$, denoting the sets of all the information sets corresponding to each player by

$$X^a = \{ x \in X \mid m(x) = a\} \text{ and } X^b = \{ x \in X \mid m(x) = b\}.$$

The extensive-form payoff functions are given by

$$F^a : Z \to \mathbf{R} \text{ and } F^b : Z \to \mathbf{R}.$$

A strategy of player a is a mapping from information sets into available choices. The strategy set for a is

$$S^a = \times_{x \in X^a} C(x)$$

and S^b is similarly defined. Let $S = S^a \times S^b$. A strategy profile $s \in S$ determines a path through the tree. Let us define the function grouping the strategy profiles according to their end points, $g : S \to S$, putting $g(s) = z$ if and only if s reaches the terminal node z. As consequence, the strategic form information sets are given by

$$f^a = F^a \circ g \text{ and } f^b = F^b \circ g.$$

In what follows, we start by taking $C(O) = \{I, 2, \ldots, K\}$. For each $k \in C(O)$, let us denote

$$S_k^a = \underset{\substack{x \in X^a \\ k \prec x}}{\times} C(x).$$

Obviously, if $m(O) = a$, then

$$S^a = C(O) \times \left(\underset{k=1}{\overset{K}{\times}} S_k^a \right).$$

Also, let us denote by $S^a(k) = \{k\} \times S_k^a$. If $m(O) \neq a$, then

$$S^a = \left(\underset{k=1}{\overset{K}{\times}} S_k^a \right).$$

We put $pr_{S_k^a} s^a = s_k^a$ if $s^a = (j, s_1^a, \ldots, s_k^a, \ldots, s_K^a)$.

Let us discuss about the step number a of a game and denote by C_{a-1} the set of all the nodes eventually reached at step $a - 1$. The elements of C_{a-1} are terminal nodes for step $a - 1$ and initial nodes for step a.

Definition 1.2. *A game is said to be tube shaped if for every step* a *and every two nodes* $x, y \in C_{a-1}$, *the tube condition* $C(x) = C(y)$ *is satisfied.*

Let us suppose that the player a choses at steps $a \in N_a$ and b at steps $ß \in N_b$, where $N_a \cup N_b = \{1, 2, 3, \ldots\}$ is the set that counts the successive choices of the two players during the game. It is obvious that if $N_a = \{1, 3, \ldots\}$ then $N_b = \{2, 4, \ldots\}$. It is easy to prove that the tube condition has as a consequence the following:

Property 1.3. *If a game is tube shaped then both* S^a *and* S^b *are rectangular, meaning that they satisfy.*

$$S^a = \underset{a \in N_a}{\times} C_a \quad \textit{and} \quad S^b = \underset{a \in N_b}{\times} C_a.$$

The existence theorem of a saddle point with respect to each rectangular subset of a tube shaped perfect information game, which is the starting point of our investigation, is the following one.

Theorem 1.4. *Fix a perfect information game and let sets* $Y^a \subseteq S^a$ *and* $Y^b \subseteq S^b$ *be rectangular. There is a saddle point of* f^a *(respectively* f^b*) with respect to* $Y^a \times Y^b$*, i.e.*

$$\max_{s^b \in Y^b} \min_{s^a \in Y^a} f^a(s^a, s^b) = \min_{s^a \in Y^a} \max_{s^b \in Y^b} f^a(s^a, s^b) \ ,$$

$$\max_{s^b \in Y^b} \min_{s^a \in Y^a} f^b(s^a, s^b) = \min_{s^a \in Y^a} \max_{s^b \in Y^b} f^b(s^a, s^b).$$

A purification result is now necessary in order to reduce the size of the solution of a procedure of finding the saddle point for a tube shaped perfect information game, computed by an algorithm following the steps of the proof of the theorem 1.4.

2. PURIFICATION RESULT

First of all we prove that a row is inadmissible with respect to a rectangular subset only if it is weakly dominated by a combination that reaches a single subtree k.

Lemma 2.1. *For a perfect information game with* $m(O) = a$*, if the strategy* s^a *is inadmissible with respect to a rectangular set* Y^b *then there exists* k *and* $\sigma^a \in \Delta(S^a(k))$ *such that*

$$f^a(\sigma^a, s^b) \geq f^a(s^a, s^b), \text{ for every } s^b \in Y^b,$$

$$f^a(\sigma^a, s^b) > f^a(s^a, s^b), \text{ for some } s^b \in Y^b.$$

Proof. Let $\hat{s}^a \in \Delta(S^a)$ be such that

$$f^a(\hat{s}^a, s^b) \geq f^a(s^a, s^b), \text{ for every } s^b \in Y^b,$$

$$f^a(\hat{s}^a, s^b) > f^a(s^a, s^b), \text{ for some } s^b \in Y^b.$$

Let $k = 1, 2, \ldots, \overline{K}$, where $\overline{K} \leq K$, be chosen such that $\hat{s}^a(r^a) > 0$ for some $r^a \in S^a(k)$. For each $k = 1, 2, \ldots, \overline{K}$, define

$$\hat{s}^a_k(r^a) = \frac{\hat{s}^a(r^a)}{\sum_{q^a \in S^a(k)} \hat{s}^a(q^a)}.$$

Obviously, $\hat{s}^a_k \in \Delta(S^a(k))$. Further, for each $s^b \in Y^b$,

$$f^a(\hat{s}^a, s^b) = \sum_{k=1}^{\overline{K}}\left[\left(\sum_{q^a \in S^a(k)} \hat{s}^a(q^a)\right)\cdot f^a(\hat{s}^a_k, s^b)\right].$$

Suppose that, for each $k = 1, 2, \ldots, \overline{K}$, $\hat{s}^a_k$ does not weakly dominate s^a with respect to Y^b. Then, for each such k, only one of the following situations can hold:

(i) $f^a(\hat{s}^a_k, s^b) = f^a(s^a, s^b)$ for each $s^b \in Y^b$,

(ii) there exists $r^b \in Y^b$ with $f^a(s^a, r^b) > f^a(\hat{s}^a_k, r^b)$.

Let us notice that if (i) holds for all $k = 1, 2, \ldots, \overline{K}$, then for all $s^b \in Y^b$

$$f^a(\hat{s}^a, s^b) = \sum_{k=1}^{\overline{K}}\left[\left(\sum_{q^a \in S^a(k)} \hat{s}^a(q^a)\right)\cdot f^a(s^a, s^b)\right] = f^a(s^a, s^b),$$

which is a contradiction. So, (ii) must hold for some k.

For each $k = 1, 2, \ldots, \overline{K}$ satisfying (ii), let

$$r^b_k := pr_{S^b(k)} r^b$$

and, if $s^a \in S^a(j)$ for $j \neq k$, let $r^b_j := pr_{S^b(j)} r^b$. Set $r^b = (r^b_1, \ldots, r^b_{\overline{K}})$ where r^b_k is as above, if defined, and otherwise r^b_k is an arbitrary element of Y_k. Since Y^b is rectangular, $r^b \in Y^b$. But

$$f^a(\hat{s}^a, s^b) = \sum_{k=1}^{\overline{K}}\left[\left(\sum_{q^a \in S^a(k)} \hat{s}^a(q^a)\right)\cdot f^a(s^a, s^b)\right] < f^a(s^a, s^b),$$

where the inequality comes from the fact that (ii) holds for some $k = 1, 2, \ldots, \overline{K}$. ♦

The required purification theorem is now easy to prove, by a forward looking procedure.

Theorem 2.2. *Let us consider a tube perfect information game. If the strategy* s^a *is inadmissible with respect to a rectangular set* $Y^b \subseteq S^b$, *then there exists* $r^a \in S^a$ *such that*

$$f^a(r^a, s^b) \geq f^a(s^a, s^b), \textit{ for every } s^b \in Y^b,$$

$$f^a(r^a, s^b) > f^a(s^a, s^b), \textit{ for some } s^b \in Y^b.$$

Proof. As in the case of theorem 1.4, we proceed by induction on the length of the tree.

Length = 1. Let us suppose $m(O)=a$, without loss of generality. Here $S^b=\{\varnothing\}$, implying that s^b is admissible. Moreover, if there exists $\sigma^a \in \Delta(S^a)$ such that

$$f^a(\sigma^a, O) > f^a(s^a, O)$$

then there is $r^a \in \mathrm{Supp}\,\sigma^a$ such that $f^a(\sigma^a, O) > f^a(s^a, O)$, as required.

Length ³ 2. Assume that the theorem is true for any tree of length ? or less and fix a tree of length $? +1$.

First, suppose that $m(O) = a$. Fix $s^a \in S^a(1)$ and suppose that s^a is inadmissible with respect to a rectangular set Y^b. Then, by lemma 2.1, there exists a number k with $\sigma^a \in \Delta(S^a(k))$ such that, for all $s^b \in Y^b$. Without loss of generality, let us pick σ^a such that $r^a \in \mathrm{Supp}\,\sigma^a$ implies that there is $s^b \in Y^b$ with

$$f^a(r^a, s^b) \neq f^a(s^a, s^b).$$

Suppose first that $\mathrm{Supp}\,\sigma^a \subseteq S^a(1)$. Then there is $r^a \in S^a(1)$ such that

$$f^a(r^a, s_1^b) \geq f^a(s^a, s_1^b), \text{ for every } s_1^b \in Y^b,$$

$$f^a(r^a, s_1^b) > f^a(s^a, s_1^b), \text{ for some } s_1^b \in Y^b,$$

by the induction hypothesis. Now the result is obvious.

Let now $\mathrm{Supp}\,\sigma^a \subseteq S^a(k)$, for $k \neq 1$. Pick

$$r_1^b \in \arg\max_{s_1^b \in Y_1} f^a(s^a, s_1^b)$$

and let us denote

$$Y_0^b = \{s^b \in Y^b \mid \mathrm{pr}_{Y_1^b} s^b = r_1^b\}.$$

Let $Y_0^a := \mathrm{Supp}\,s$ and denote

$$f^a(\bar{s}^a, \bar{s}^b) = \max_{Y_0^b} \min_{Y_o^a} f^a(\cdot,\cdot),$$

$$f^a(\underline{s}^a, \underline{s}^b) = \min_{Y_0^b} \max_{Y_o^a} f^a(\cdot,\cdot).$$

Then, for all $s^b \in Y^b$,

$$f^a(s^a, s^b) \leq f^a(s^a, \underline{s}^b) \leq f^a(\underline{s}^a, \underline{s}^b) = f^a(\bar{s}^a, \bar{s}^b) \leq f^a(\bar{s}^a, s^b),$$

where the first inequality comes from the choice of r_1^b, the second one from the definition of min max, the third one from theorem 1.4 and the last one from the definition of maxmin. The choice of s^b leads to the existence of $s^b \in Y^b$ with

$$f^a(\bar{s}^a, s^b) > f^a(s^a, s^b),$$

as required.

Now, suppose $m(O) \neq a$. Let $\sigma^a \in \Delta(S^a)$ be such that, for all the strategies $s^b \in Y^b$, one has

$$f^a(\sigma^a, s^b) \geq f^a(s^a, s^b),$$

with a strict inequality for some $s^b \in Y^b$. Let us denote by $s^a = (s_1^a, \ldots, s_K^a)$. For each k having the property that $Y^b \cap S^b(k) \neq \emptyset$, define

$$s_k^a(r_k^a) = \sum_{r^a \in S^a : pr_{s_k^a} r^a = r_k^a} s^a(r^a).$$

It is readily verified that $s_k^a \in \Delta(S_K^a)$. For each k with $Y^b \cap S^b(k) \neq \emptyset$,

$$f^a(s_k^a, s^b) \geq f^a(s_k^a, s^b), \text{ for all } s^b \in Y^b \cap S^b(k).$$

Moreover, there exists k, with $s^b \in Y^b \cap S^b(k)$, such that

$$f^a(s_k^a, s^b) > f^a(s_k^a, s^b).$$

By the induction hypothesis, for each such k, there exists $r_k^a \in S^a(k)$ with $f^a(r_k^a, s^b) \geq f^a(s_k^a, s^b)$, for every $s^b \in Y^b \cap S^b(k)$ and strict inequality for some $s^b \in Y^b \cap S^b(k)$. For all other k, set $r_k^a = s_k^a$. Let us denote by $r^a = (r_1^a, \ldots, r_K^a)$; it is readily verified that r^a satisfies the required property. ♦

Among the games satisfying the tube condition one can find the Japanese go and the well-known draughts. In fact, each perfect information game that develops in conditions of equal opportunities for the players is a tube shaped game. Also, in economics, in investment planning and in marketing one can find problems needing the above-presented procedure.

3. REFERENCES

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