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PERIODIC SOLUTION WITH AN APPROXIMATE METHOD FOR NONLINEAR OSCILLATIONS

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Abstract: The rational harmonic balance method is employed to study the periodic solution of a one-degree of freedom system with strongly nonlinearities. Comparisons are made with the solutions obtained by using the elliptic perturbation method or Lindstedt-Poincare method, to show the efficiency of the present method.

1. INTRODUCTION

Many researchers have developed various approximate methods for the non-autonomous non-linear oscillations. Interesting analysis has been reported by Mickens and Semwogerere [1], recommending a rational function

$$x(t) ? \frac{A\cos ? t}{1 ? B\cos 2 ? t} \tag{1}$$

for the non-linear one-dimensional oscillator differential equation

$$\ddot{x} ? f(x) ? 0 \tag{2}$$

where f(x) is an analytic function of x and $x(0) ? x_0 ? 0, \dot{x}(0) ? 0$. Here ? is the angular frequency, x_0 is the maximum amplitude and overdots denote differentiation with respect to time t. They have examined a particular case of the function $f(x)=x^3$ to conclude that the form of the solution (1) provides an excellent approximation to the actual solution of the equation (2) which is, however, true for $f(x)=x^3$ or odd functions. When f(x) is not an odd function, the approximate periodic solution (1) for the equation of motion (2) needs a modification.

Particularly interesting in the response of the Duffing oscillator to a harmonic excitation in the presence of viscous damping, which has been found to exhibit, among other features, hysteretic and chaotic behaviors. Thus, we consider a system governed by a non-dimensional differential equation of the form

$$\ddot{x} ? c\dot{x} ? ax ? bx^3 ? d ? e\cos ? t, x(0) ? x_0, \dot{x}(0) ? 0 \tag{3}$$

where a,b,c,d,e are constant parameters.

For d=e=0 into equation (3) Jacobian elliptic functions can be adopted in order to exactly solve of this equation. Soudack and Barkhan [4] were the first to use Jacobian elliptic functions to construct an approximate solution for the equation:

$$\ddot{x} ? ? x ? ? x^3 ? ? f(x, \dot{x}, t) ? 0 \tag{4}$$

such that

$$x(t) ? A(t)cn(? t ? ?(t), k) \tag{5}$$

where $cn(?, k)$ is the cosine Jacobian elliptic function and A(t), ?, f and k are called the amplitude, the angular frequency, phase and the modulus of the elliptic function respectively. Chen and Cheung [5] have developed an elliptic perturbation method for calculating periodic solutions of strongly nonlinear oscillators (4) in which the Jacobian elliptic functions are employed instead of usual circular function in the conventional perturbation procedure.

In this procedure, the rational harmonic balance method is presented for solving the oscillator of the form (3).

2. THE SOLUTION OF EQUATION (3)

The equation (3) after the transformation $?t ? ?$, becomes:

$$?^2 x'' ? ?cx' ? ax ? bx^3 ? d ? e\cos ? \tag{6}$$

where $x' ? \frac{dx}{d?}$.

For equation (6), with the initial condition $\dot{x}(0) ? 0$, we propose the solution in the form:

$$x(?) ? \frac{A ? B\cos ? ? C(2\sin ? ? \sin 2?) ? D\cos 2?}{1 ? E\cos 3?} \tag{7}$$

where A,B,C,D,E will be determined in the following. After the use of trigonometric identities and application of the method of rational harmonic balance to retain only constant terms involving, $\cos ?, \cos 2?$ $\sin ?, \sin 2?$ we obtain equations:

$$\frac{A ? B ? D}{1 ? E} ? x_0$$

$$\frac{9}{2}?^2 AE(1 ? 2E) ? a\left(1 ? \frac{E^2}{2}\right)A ? b\left(A ? \frac{3}{2}A(B^2 ? C^2 ? D^2) ? 3BC^2 ? \frac{3}{4}B^2 D ? 3C^2 D\right) ? d ? \frac{3}{2}dE^2 ? 0$$

$$?^2\left(? B ? \frac{7}{2}DE ? 4BE^2\right) ? c?(2C ? CF ? CE^2) ? a\left(B ? \frac{1}{2}BE^2 ? DE\right) ? b\left(\frac{3}{4}B^3 ? \frac{9}{2}BC^2 ? \frac{3}{2}BD^2 ?\right.$$

$$\left.3A^2 B ? 6AC^2 ? 3ABD\right) ? e\left(1 ? \frac{3}{2}E^2\right) ? 0$$

$$?^2\left(? 2C ? \frac{1}{2}CF ? 14CE^2\right) ? c?\left(\frac{7}{2}DE ? B ? \frac{1}{2}BE^2\right) ? a(2C ? CE ? CE^2) ? b\left(3A^2 C ? 9C^3 ? \frac{3}{2}B^2 C ?\right.$$

$$\left.? 3CD^2 ? 3ABC ? 6ACD\right) ? 0$$

$$?^2\left(? 4D ? \frac{1}{2}BE ? \frac{23}{2}DE^2\right) ? c?(2C ? 5CE ? CE^2) ? a\left(D ? \frac{1}{2}DE^2 ? BE\right) ? b\left(3A^2 D ? \frac{3}{2}AB^2 ?\right.$$

$$\left.6AC^2 ? \frac{3}{4}AD^3 ? \frac{3}{2}AB^2 D ? \frac{15}{4}AC^2 D\right) ? e\frac{E}{2}\left(3 ? \frac{E^2}{4}\right) ? 0$$

$$?^2\left(4C ? CE ? \frac{23}{2}CE^2\right) ? c?\left(\frac{5}{2}BE ? 2D ? DE\right)^2 ? a\left(c ? 2CE ? \frac{1}{2}CE^2\right) ? b\left(\frac{9}{4}CD^2 ? \frac{27}{4}C^3 ?\right.$$

$$\left.? \frac{3}{2}B^2 C ? 3AC ? 6ABC\right) ? 0 \tag{8}$$

$$9?^2 AE ? 2aE(A ? B ? D) ? b\left(\frac{B^3}{4} ? \frac{15}{4}BC^2 ? \frac{3}{4}BD^2 ? 3C^2 D ? 6AC^2 ? 3ABD\right) ? 3dE\left(1 ? \frac{E^2}{4}\right) ? 0$$

The equations (8) are solved numerically using the Newton-Raphon's iterative procedure and Levenberg-Marquardt algorithm.

To verify the adequacy of the proposed approximate periodic solution (7) for Duffing equation, the following cases have been examined:

Case 1: For a=1, c=0, d=e=0, in [8] is obtained (with Modified Homotopy Perturbation Method):

$$? ? \sqrt{\frac{1}{2}?1 ? \frac{3}{4} b x_0^2 ?? \frac{1}{2}\sqrt{1 ? \frac{3}{2} b x_0^2 ? \frac{15}{32} b^2 x_0^4}} \qquad (9)$$

Table 1 shows the results for this case and it can be seen that the solution (8) for ? is very close to the exact solution as noticed in all the above cases.

Table 1:Comparisons of the results in case 1

bx_0^2	? [8]	? [present study]	? [exact solution]
1	1,392	1,395	1,39214
10	3,2331	3,256	3,13287
100	9,4102	9,4214	9,379943

Case 2 For a=-0,5; b=0,5; c=0,1; d=0, e=0,4; $x_0 ? ?1$, $? ? 3$, in [9] is find the following approximate solution:

$$x(t) ? ?0,9981 ? 0,049899 \cos 3t ? 0,0018703 \sin 3t \qquad (10)$$

Comparisons with our method in the given initial conditions, formula (7) leads to excellent approximations:

$$x(t) ? \frac{?0,972043 ? 0,049488 \cos 3t ? 0,001855 \sin 3t ? 0,00029 \cos 6t}{1 ? 0,0003 \cos 9t} \qquad (11)$$

or

$$x(t) ? ?0,972043 ? 0,049488 \cos 3t ? 0,001855 \sin 3t \qquad (12)$$

Case 3: For a=-0,5; b=0,5; c=0,1; d=0; e=0,4; $x_0 ? ?1$, $? ? 2{,}1$ în [9] is find the approximate solution:

$$x(t) ? ?0,90995 ? 0,11609 \cos 2{,}1t ? 0,0071020 \sin 2{,}1t \qquad (13)$$

Our method leads to the solution:

$$x(t) ? ?0,900242 ? 0,108273 \cos 2{,}1t ? 0,006155 \sin 2{,}1t \qquad (14)$$

This agrees with the results obtained in [9].

3. CONCLUSIONS

In the present paper, the rational harmonic balance method is given for non-linear vibrations. This method can be widely used to solve non-linear vibration problems, and can also be used to study the stability of the solution. Moreover, it has good convergence and accuracy.

REFERENCES

1. R.E.Mickens, D.Semwogerere, Fourier analysis of a rational harmonic balance approximaton for periodic solutions, Journal of Sound and Vibration, 195, pp.528-530 (1996)
2. R.E.Mickens, A generalization of the harmonic balance, Journal of Sound and Vibration, 111, pp.515-518 (1986)
3. A.N.Nayfeh, D.T.Mook, Nonlinear oscillations, N.Y.-Willey-Interscience (1979)
4. A.C.Soudack, P.G.Barkhan, On the transient solution of the unforced Duffing equation with large damping, Int.J. of Control, 13, pp.767-769 (1971)
5. S.H.Cheung, Y.K.Cheung, *An elliptic perturbation method for certain strongly non-linear oscillators*, J. of Sound and Vibration, 192, pp.935-937 (2000)
6. R.S.Sarma, B.N.Rao, A rational harmonic balance approximation for the Duffing equation of mixed parity, Journal of Sound and Vibration, 207, pp.597-599 (1997)
7. V.Marinca, N.Herisanu, *On the rational harmonic balance method for mixed parity non-linear oscillators*, Analele Univ. A. Vlaicu Arad, pp.230-235 (2004)
8. V.Marinca, *Application of Modified Homotopy Perturbation Method to non-linear oscillators* (to be published)
9. X.M.Tian, L.J.Feng, C.Delin, Non-linear vibration by a new method, Journal of Sound and vibration, 215, pp.475-487 (1998)