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STUDY OF THE STABILITY OF PERIODIC RESPONSES FOR THE DUFFING OSCILLATOR

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Abstract: The response of a damped Duffing oscillator of the softening type to a harmonic excitation is analyzed, by means of the rational harmonic balance method. Floquet analysis is used to study the stability of the solution and is applied over the period T as the coefficients in variational equation have a period T.

1. INTRODUCTION

Non-linear vibratory systems under harmonic excitation have been studied extensively for a few decades by means of approximate analytical methods and their regular behavior has been described and explained in terms of now classical concepts of principal and secondary resonances (sub-, ultra- or subultraharmonic ones), resonances curves, stability limits of particular approximate solutions and jump phenomena.

A variety of analytical methods have been applied to stability analysis of periodic systems. They include Hill's infinite determinants method, various perturbation methods, the averaging method and Floquet theory combined with numerical integration. It is well established that Hill's method is not convenient for numerical computations, particularly in the case of high order systems, because it is based on computing determinations of large matrices (theoretically of infinite order) to determine transition point between stable and unstable regions in the parameter space. Perturbation methods are also limited in application since they are based on expanding the solution in terms of a small parameter that multiplies the periodic terms. Averaging methods are also restricted to systems that possess small parameters and slowly varying amplitudes and suffer from similar drawbacks in accuracy as those of perturbation methods. Different methods have been proposed to study the stability of periodic solutions using numerical integration. These methods are based on approximating the period function by special functions, for example step functions, Fourier expansion, or Chebyshev polynomials. An alternative integration procedure usually employs Runge-Kutta or Hamming's methods to obtain the transition matrix from which stability conditions are derived numerically. The limitation of this approach is that analytical dependence on parameters is difficult to obtain and, therefore, it is usually not computed.

In this paper, the rational harmonic balance method (RHBM) is used to determine the equations that describe the approximate periodic solution of the system in the given initial conditions. The Floquet-Hill method for the study of the solution of Duffing oscillator is used.

2. THE METHOD OF ANALYSIS

The governing equation considered can be expressed as the following form:

$$\ddot{x} + c\dot{x} + ax + bx^3 = d + e\cos\omega t \qquad (1)$$

with the initial conditions:

$$x(0) = x_0, \dot{x}(0) = 0 \quad (2)$$

where a,b,c,d,e are constant parameters.

To ascertain the stability of the periodic solution x(t), we examine the time evolution of the solution after the application of an infinitesimal arbitrary disturbance y(t) in the form:

$$\bar{x}(t) = x(t) + y(t) \quad (3)$$

The stability of x(t) then depends on whether y(t) grows or decays with t. Substituting equation (3) into equation (1) and keeping linear terms in y(t), we obtain:

$$\ddot{y}(t) + c\dot{y}(t) + (a + 3bx^2(t))y(t) = 0 \quad (4)$$

which is a linear ordinary differential equation with periodic coefficients having the period $T = \frac{2\pi}{\omega}$. The existence of non-trivial solution can be shown via Floquet's theorem, which calls for solutions of the form:

$$y(t + T) = sy(t) \quad (5)$$

where s is an eigenvalue (also called a Floquet multiplier) of the monodromy matrix M whose elements are associated with equation (4) through the relations:

$$y_1(t + T) = m_{11}y_1(t) + m_{12}y_2(t) \quad (6)$$
$$y_2(t + T) = m_{21}y_1(t) + m_{22}y_2(t) \quad (7)$$

where m_{ij} are constants. The functions $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of equation (4). To generate y_1 and y_2, we use the initial conditions:

$$y_1(0) = 1; \dot{y}_1(0) = 0 \quad (8)$$
$$y_2(0) = 0; \dot{y}_2(0) = 1 \quad (9)$$

The solution x(t) is a stable orbit provided that y(t) does not grow with t. This requires that

$$|s| < 1 \quad (10)$$

that is, the eigenvalue of M must remain inside the unit circle in the complex plane.

The monodromy matrix M can be obtained by means of MHPM [8] subject to the initial conditions (8) and (9). It follows from equations (6)-(9) that

$$M = \begin{bmatrix} y_1(T) & \dot{y}_1(T) \\ y_2(T) & \dot{y}_2(T) \end{bmatrix} \quad (11)$$

Therefore

$$s^2 - [y_1(T) + \dot{y}_2(T)]s + [y_1(T)\dot{y}_2(T) - \dot{y}_1(T)y_2(T)] = 0 \quad (12)$$

The values of s determine the stability of the approximate solution x(t) according to equation (10). The manner in which the eigenvalue s leaves the unit circle characterizes the local qualitative (bifurcations) occurring to the orbit. For the dissipative one-degree-of-freedom system described by equation (1) there are two ways in which s can leave the unit circle [2] each of which creates independent patterns of instability in the periodic orbit. An eigenvalue can leave the unit circle through the real axis at either -1 or +1. It follows from equation (1) that the period of x(t) is $T ? \frac{2?}{?}$. Consequently, when s leaves the unit circle through -1, y(t+2T)=y(t) according to equation (5) and hence solution with the period 2T is stable, indicating a period-doubling of flip bifurcation. On the other hand, when s leaves the unit circle through +1, equation (5) indicates that y(t+T)=y(t), which implies the coexistence of a stable and an unstable attractor with the period T. The result is a saddle-mode or tangent bifurcation, which results in a jump in the response of the system.

3. EXAMPLE

Consider the following Duffing system:

$$\ddot{x} ? 0{,}1\dot{x} ? 0{,}5x ? 0{,}5x^3 ? 0{,}4\cos ?t \qquad (13)$$

From now on, it is assumed that $T ? \frac{2?}{?}$. In our computation, a T-periodic solution is found. Assuming that ?=3, and applying the method RHBM [8], after the use of trigonometric identities and for $x(0) ? x_0 ? ?1; \dot{x}(0) ? 0$, one obtains:

$$x(t) ? ?0{,}972043 ? 0{,}049488\cos 3t ? 0{,}001855\sin 3t \qquad (14)$$

For equation (13) with the solution (14), equation (4) becomes

$$\ddot{y} ? 0{,}1\dot{y} ? (0{,}919141 ? 0{,}144313\cos 3t ? 0{,}005409\sin 3t)y ? 0 \qquad (15)$$

For equation (15) we propose with RHBM the solution in the form

$$y(t) ? \frac{A ? B\cos 1{,}5? ? C\sin 1{,}5? ? D\cos 3? ? E\sin 3t}{1 ? F\cos 6?} \qquad (16)$$

The solution $y_1(t)$ obtained for initial conditions (8) is find as

$$y_1(t) ? 0{,}519 ? 0{,}409\cos 1{,}5t ? 0{,}121\sin 1{,}5t ? 0{,}072\cos 3t ? 0{,}060\sin 3t \qquad (17)$$

The solution $y_2(t)$ obtained for initial conditions (9) is find as

$$y_2(t) ? 0{,}368 ? 0{,}306\cos 1{,}5t ? 0{,}501\sin 1{,}5t ? 0{,}062\cos 3t ? 0{,}082\sin 3t \qquad (18)$$

The monodromy matrix (11) becomes

$$M = \begin{bmatrix} 0{,}182 & 0{,}612 \\ -0{,}361 & -0{,}505 \end{bmatrix} \qquad (19)$$

Equation (12) is of the form:

$$s^2 + 0{,}3235\,s + 0{,}129237 = 0 \qquad (20)$$

with the solution (complex-conjugates):

$$s_1 = -0{,}16175 + 0{,}3210513\,i \quad ; \qquad s_2 = -0{,}16175 - 0{,}3210513\,i \quad ; \qquad i = \sqrt{-1} \qquad (21)$$

Therefore $|s_1| = |s_2| = 0{,}359495 < 1$. It follows that the solution (14) of the equation (13) is stable.

4. CONCLUSIONS

In the present paper, RHBM approach is used for non-linear vibration. The stability of the solution is studied with the Floquet theory.

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