LARGE-SCALE DYNAMICAL SYSTEMS AND PSEUDOSPECTRA

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Abstract. In this paper we will deal with the problem of model reduction of large-scale dynamical systems and the related pseudospectral aspects.The key to reducing a system from a high order to very low order depends on the decay of the Hankel singular values. For a system with diagonalizable *A* Hankel singular values depends on the condition number of the matrix whose columns are the normalized eigenvectors Since large pseudospectra are indicative of highly non-normal matrices, examining psesudospectra naturally gives us insight into whether we should expect to be able to get a decent reduction. The ability to reduce a system well depends on well-behaved pseudospectra.

1. INTRODUCTION

Model reduction has a long history in the systems and control literature. In fact, the general topic of dimension reduction in dynamical systems is pervasive in the applied mathematics literature. The large system dimension makes the computation infeasible due to memory, time limitations and ill-conditioning. One approach to overcoming this is through model reduction. The goal is to produce a low dimensional system that has similar response characteristics as the original system with far lower storage requirements and evaluation time. The resulting reduced model might be used to replace the original system as a component in a larger simulation or it might be used to develop a low dimensional controller suitable for real time applications.

A linear time-invariant dynamical system can be represented by means of an integral (convolution) operator; if, in addition, is finite-dimensional, then this operator has finite rank and is hence compact. Consequently, it has a set of finitely many non-zero singular values. Thus, in principle, the SVD approximation method can be used to simplify such dynamical systems. Indeed, there is a set of invariants called the Hankel singular values (HSV) which can be attached to every linear, constant, finite-dimensional system. These invariants play the same role for dynamical systems as the singular values play for constant finite-dimensional matrices. In other words, they determine the complexity of the reduced system and at the same time they provide a global error bound for the resulting approximation. This gives rise to two model reduction methods, namely Balanced model reduction and Hankel norm approximation. Many have observed that the Hankel singular values of many systems decay extremely rapidly. Hence very low rank approximations are possible and accurate low-order reduced models will result.

One model reduction scheme that is well grounded in theory and most commonly used is the so-called Balanced Model Reduction first introduced by Mullis and Roberts [11] and later in the systems and control literature by Moore [10]. To apply balanced reduction, first the system is transformed to a basis where the states which are difficult to reach are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the controllability and the observability gramians, which are solutions to the controllability and the observability Lyapunov equations. Then, the reduced model is obtained by truncating the states which have this property. We will call this *the Lyapunov balancing method*. When applied to stable systems, Lyapunov balanced reduction preserves stability [12] and provides a bound on the approximation error [7].

Besides the Lyapunov balancing method, other types of balancing exist such as stochastic balancing, bounded real balancing, positive real balancing, LQG balancing and frequency weighted balancing. All the balancing techniques mentioned above try to approximate the full-order model $G(s)$ over all frequencies. However, in many applications one is only interested in a given frequency interval. In these cases *the frequency weighted balanced reduction* is used which tries to reduce the error between $G(s)$ and $\hat{G}(s)$ over the specified frequency range, i.e. the weighted error. Several ways of weighted balancing have been introduced in the literature. Lyapunov balanced reduction was extended to the frequency weighted balanced reduction by Enns [7].

2. MODEL REDUCTION

Consider the linear time-invariat (LTI) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t),\ x(0) = x_0 \end{aligned} \tag{1}$$

where $A \in \mathrm{R}^{n\times n}$ is the state matrix, $B \in \mathrm{R}^{n\times m}$, $C \in \mathrm{R}^{p\times m}$, and $x_0 \in \mathrm{R}^n$ is initial state of the system. Here, *n* is the order (or the state space dimension) of the system. The associate transfer function matrix (TFM) obtained from taking Laplace transforms in (1) and assuming $x_0 = 0$ is

$$G(s) = C(sI_n - A)^{-1}B + D \tag{2}$$

In model reduction we are faced with the problem of finding a reduced-order LTI system

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t) \\ \hat{y}(t) &= \hat{C}\hat{x}(t) + \hat{D}\hat{u}(t),\ \hat{x}(0) = \hat{x}_0 \end{aligned} \tag{3}$$

of order $r, r << n$, and associated TMF $\hat{G}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}$ which approximate $G(s)$ such that the following properties are satisfied:

1. the approximate error $\|y - \hat{y}\|$ is small, and their exist a global error bound;
2. system properties, like stability, are preserved;
3. the methods are computationally efficient.

In assessing the quality of reduce order model, one often looks at the following characteristic of the system to be approximated:

a. the eingenvalues of *A* (or at least the closed ones to the $j\omega$ axis), which are also the poles of $G(s)$

b. the controllability Gramian $\mathcal{G}_c$ and observability Gramian $\mathcal{G}_o$ of the system which are the solution of of the Lyapunov equations

$$A\mathcal{G}_c + \mathcal{G}_c A^T + BB^T = 0\,, \quad A^T\mathcal{G}_0 + \mathcal{G}_0 A + C^T C = 0 \qquad (4)$$

c. the singular values of the Hankel map –called the Hankel singular values (HSV) which are also the square-roots of the eigenvalues of $\mathcal{G}_c\,\mathcal{G}_o$
d. the largest singular value of the transfer function as function of frequency called the frequency response

$$\sigma(\omega) = \|G(j\omega)\|_2 \qquad (5)$$

These characteristic can be compared with those of the reduced order model $\hat{G}(s)$.
Under the assumption that $G(s)$ is asymptotically stable and minimal, the above equations (4) have unique symmetric positive definite solutions $\mathcal{G}_c$ and $\mathcal{G}_o$,respectively.
The square roots of the eigenvalues of the product $\mathcal{G}_c\,\mathcal{G}_o$ are so-called Hankel singular values $\sigma_i(G(s))$ of the system $G(s)$

$$\sigma_i(G(s)) = \sqrt{\lambda_i(\mathcal{G}_c\mathcal{G}_o)}\,. \qquad (6)$$

It is easy to see that , $\sigma_i(G(s))$ are basis independent. In many case, the eigenvalues of $\mathcal{G}_c, \mathcal{G}_o$ as well as the Hankel singular values decay very rapidly [1].
Under the state transformation $\hat{x} = Tx$, the two Gramians are transformed by congruence: $\hat{\mathcal{G}}_c = T\mathcal{G}_c T^T$, $\hat{\mathcal{G}}_o = T^{-T}\mathcal{G}_o T^{-1}$, thus $\lambda_i(\mathcal{G}_c\mathcal{G}_o)$ are input-output invariants of systems. These are fundamental invariants which determine how well the system $G(s)$ can be approximated by reduced-order system $\hat{G}(s)$. They play the same role for dynamical systems that the singular values play for finite- dimensional matrices. Their computation is therefore of primary importance.
First, we must balance the system. This simply means we make the states x(t) of the system which are difficult to reach also difficult to observe. We do not want to eliminate states that are hard to reach but strongly observable, or vice versa. We obtain an equivalent state-space system that is balanced by the methods described in [2], [3], [15], or by using the MATLAB command balreal on a stable, minimal state-space representation sysm: >> sysb = balreal(sysm). Once we have a balanced representation, we can perform a balanced truncation of the system by examining the Hankel singular values of the system. We can retrieve those values by giving an extra output argument *g* to the command balreal: >> [sysb,g] = balreal(sysm). The hope behind balanced truncation is that we can simply delete the states associated with small singular values and still retain accurate estimates of the output. To delete the states associated with the singular values with indices stored in a vector ELIM, use the Control Toolbox command >> sysr = modred(sysb, ELIM).
The above sequence of commands work in general to produce a reduced model, but they are not sturdy enough to reduce a model with the prerequisite traits mathematically but not numerically. If A is not numerically stable even if it is, mathematically balreal breaks down. Solution is use the μ-Analysis and Synthesis Toolbox and the sysbal command

which produces a balanced system with Hankel singular values greater than a specified tolerance. The Control Toolbox seems to have issues with near-unstable systems, but the μ-Analysis and Synthesis Toolbox seems to be more robust.

3. PSEUDOSPECTRA AND MODEL REDUCTION

The pseudospectrum (set of pseudo-eigenvalues) is a powerful modeling tool: for example, large real parts of pseudo-eigenvalues (rather than eigenvalues themselves) often reveal the behavior of dynamical systems.
For a real ε > 0, the ε-*pseudospectrum* of *A* is the set

$$\Lambda_{\varepsilon} = \{z \in C : z \in \Lambda(X) \text{ where} \|X - A\| < \varepsilon\} \qquad (7)$$

Throughout, $\|\cdot\|$ denotes the operator 2-norm. Any element of the pseudospectrum is called a *pseudo-eingenvalue.*
If $\sigma_{\min}(A)$ denotes the smallest singular value of *A* then we have a useful caracterization of the pseudospectrum

$$\Lambda_{\varepsilon}(A) = \{z \in C : \sigma_{\min}(zI - A) \le \varepsilon\} \qquad (8)$$

Thus the pseudospectra of *A* are the sets in the *z*-plane bounded by level curves of the function $g(z) = \sigma_{\min}(zI - A) = \left\|(A - zI)^{-1}\right\|^{-1}$, where we interpret the right-hand side as zero when $z \in \Lambda(A)$.

Many classical problems of robustness and stability in control theory aim to move the eigenvalues of a parameterized matrix into some prescribed domain of the complex plane. However, a simple consideration of the spectrum alone has serious drawbacks in many contexts. As Trefethen and others have pointed out (see [13], [14], and the references therein), pseudospectra of a matrix are more informative and more robust in many modeling frameworks, and in particular as indicators of stability, of robustness [4], [5] and of controllability [9].

As we have seen, the key to reducing a system from a high order to very low order depends on the decay of the Hankel singular values. If they decay rapidly, one can find a low-order system, which can be manipulated at low cost, that tells just about as much as the original system. In the work by [1], we see that the decay rate of the Hankel singular values for a system with diagonalizable A depends on the condition number of the matrix whose columns are the normalized eigenvectors A. If the condition number is modest, then a low-order reduction is likely to be achievable. Since large pseudospectra are indicative of highly non-normal matrices, examining psesudospectra naturally gives us insight into whether we should expect to be able to get a decent reduction. So, pseudospectral implications on stability should be considered when trying to find a balanced truncation. The ability to reduce a system well depends on well-behaved pseudospectra.

4. BENCHMARK EXAMPLE

In [6], [3], provided several models for reductions. We will look at clamped beam model (BEAM). The clamped beam model has 348 states, it is obtained by spatial discretization of an appropriate partial differential equation. The input represents the force applied to the structure at the free end, and the output is the resulting displacement. For this example, the real part of the pole closest to imaginary axis is -5.05×10^{-3}. We approximate the system with a model of order 35 (tolerance 10^{-3}). The data were obtained from [6].

First, we take a look at the pseudospectra of the *A* matrix in the model in Figure 1 and Figure 2. We immediately see that the issue of stability will arise, so we abandon the Control Toolbox and go directly to the µ-Analysis and Synthesis Toolbox. We use Eigtool [16] to see that pseudospectra of A are modest (and indeed the condition number of the eigenvector matrix is modest, as expected, at a value of 358), so we expect to have respectably decaying Hankel singular values. We find that the Hankel singular values returned by sysbal decay rapidly as seen in Figure 3, but there are many of non-negligible size. This leads us to believe we can obtain a modestly low-order reduction via the strunc command, but we doubt we can obtain one of very low order. Indeed we see we can reduce the system well with only 10% of the states, Figures 4. We also note that the ε pseudospectra can extend 40ε away from an eigenvalue of *A*, so we expect transient growth of at least 40 [13].

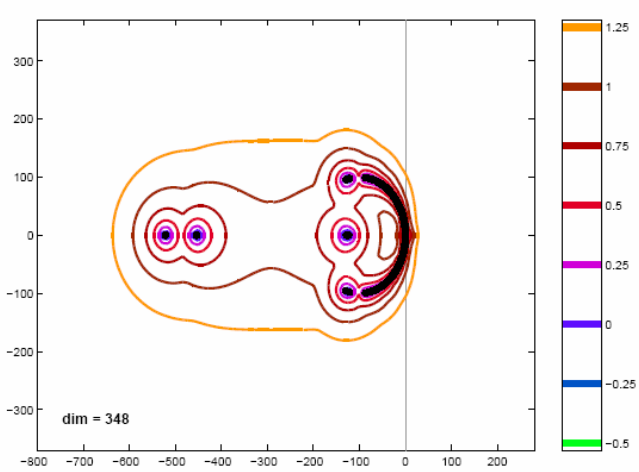

Figure 1 Pseudospectra of beam model

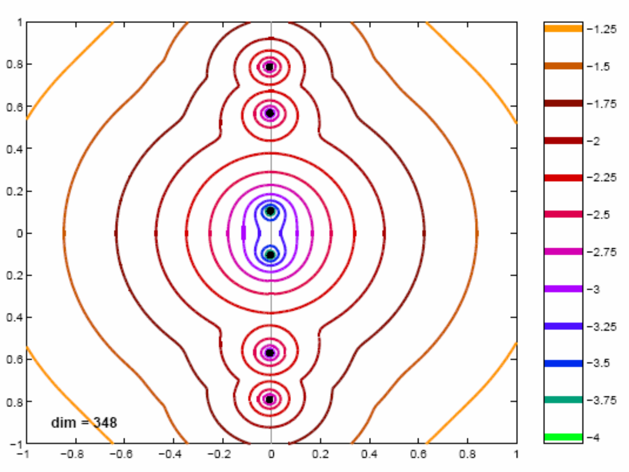

Figure 2 Pseudospectra of beam model near the imaginary axis

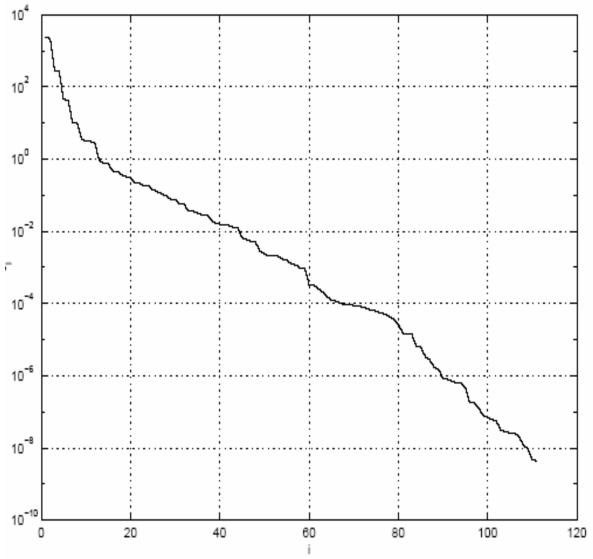

Figure 3 The Hankel singular values

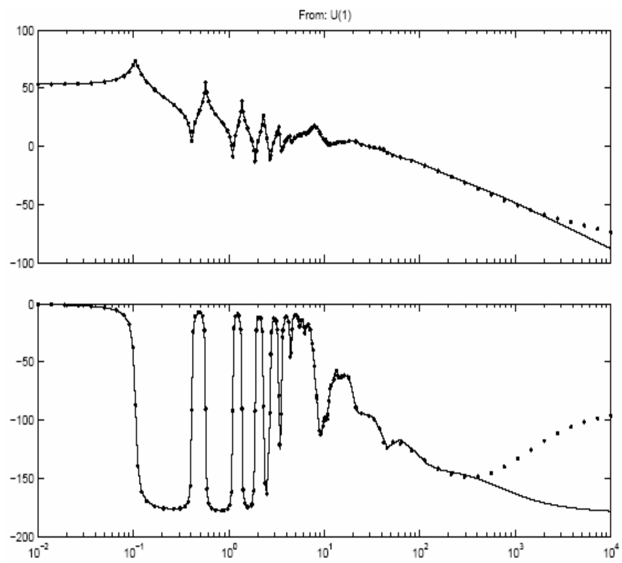

Figure 4 Reduced system with 35 states

5. CONCLUSIONS

Decay of Hankel singular values determine ability to reduce system well. Condition number of the eigenvector matrix of A governs the decay of the Hankel values.

Pseudospectra can give a good idea of that rate of decay, but perhaps not a hard-and-fast rule for exactly how well the system can be reduced.

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