ON THE STUDY OF NON-LINEAR SUSPENSION

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Abstract
In our paper we develop a model for an automotive with non-linear suspension. The non-linearity is established with a third order function for the stress-strain dependency. In our model we determined the equilibrium positions or both the hard and soft stiffness and we studied the stability of these positions

1. INTRODUCTION

We consider the automobile modeled by a two degrees of freedom model as it can be seen in Figure 1. The second stiffness is a non-linear one, and the elastic force, which appears in such an element is

$$F_{e_2} = k_2 z + \varepsilon_2 z^3, \quad (1)$$

where z is the elongation of the element, $k_2 > 0$, and ε_2 can be positive or negative. At equilibrium, one obtains the equations:

$$m_2 g = k_2 z_{20} + \varepsilon_2 z_{20}^3;\ m_1 g = k_1 z_{10} - k_2 z_{20} - \varepsilon_2 z_{20}^3. \quad (2)$$

Representing now the forces for the moving status and applying the Newton's second law, we obtain the equations:

$$\begin{aligned} m_1\ddot{z}_1 &= k_2(z_2 - z_1 - z_{20}) + \varepsilon_2(z_2 - z_1 - z_{20})^3 - k_1(z_1 - z_{10}) - m_1 g;\\ m_2\ddot{z}_2 &= -k_2(z_2 - z_1 - z_{20}) - \varepsilon_2(z_2 - z_1 - z_{20})^3 - m_2 g. \end{aligned} \quad (3)$$

From equations (2) and (3) we get:

$$\begin{aligned} m_1\ddot{z}_1 &= k_2(z_2 - z_1) + \varepsilon_2(z_2 - z_1)^3 - 3\varepsilon_2(z_2 - z_1)^2 z_{20} + 3\varepsilon_2(z_1 - z_{10})z_{20}^2 - k_1 z_1;\\ m_2\ddot{z}_2 &= -k_2(z_2 - z_1) - \varepsilon_2(z_2 - z_1)^3 + 3\varepsilon_2(z_2 - z_1)^2 z_{20} - 3\varepsilon_2(z_2 - z_1)z_{20}^2. \end{aligned} \quad (4)$$

Denoting:

$$z_1 = \xi_1;\ z_2 = \xi_2;\ \dot{z}_1 = \xi_3;\ \dot{z}_2 = \xi_4, \quad (5)$$

results the following system of four first order non-linear differential equations

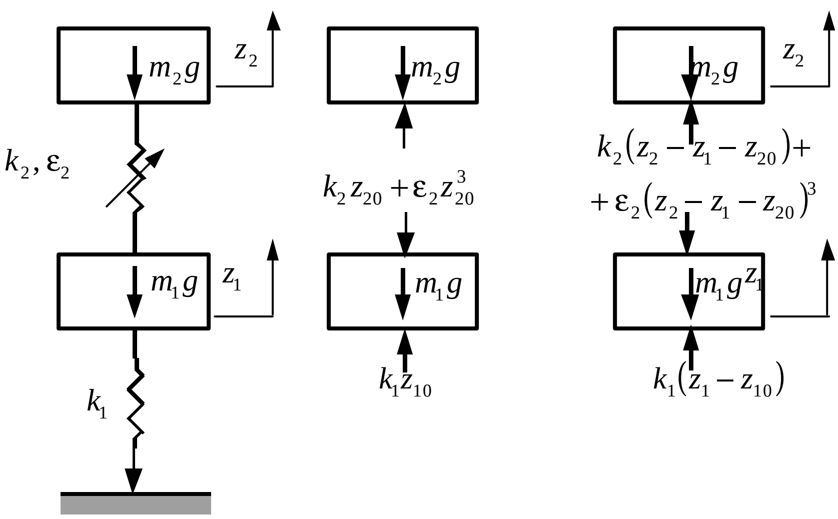

Fig. 1. Model of an automobile with non-linear suspension.

$$\frac{d\xi_1}{dt}=\xi_3\,;\;\frac{d\xi_2}{dt}=\xi_4\,;$$
$$m_1\frac{d\xi_3}{dt}=k_2(\xi_2-\xi_1)+\varepsilon_2(\xi_2-\xi_1)^3-3\varepsilon_2(\xi_2-\xi_1)^2z_{20}+3\varepsilon_2(\xi_2-\xi_1)z_{20}^2-k_1\xi_1\,; \tag{6}$$
$$m_2\frac{d\xi_4}{dt}=-k_2(\xi_2-\xi_1)-\varepsilon_2(\xi_2-\xi_1)^3+3\varepsilon_2(\xi_2-\xi_1)^2z_{20}-3\varepsilon_2(\xi_2-\xi_1)z_{20}^2\,.$$

2. ESTABLISHING THE EQUILIBRIUM POSITIONS

The equilibrium positions are at the intersections of the nullclines. From equations (6), one obtains:

$$\xi_3=0\,;\;\xi_4=0\,;$$
$$k_2(\xi_2-\xi_1)+\varepsilon_2(\xi_2-\xi_1)^3-3\varepsilon_2(\xi_2-\xi_1)^2z_{20}+3\varepsilon_2(\xi_2-\xi_1)z_{20}^2-k_1\xi_1=0\,; \tag{7}$$
$$-k_2(\xi_2-\xi_1)-\varepsilon_2(\xi_2-\xi_1)^3+3\varepsilon_2(\xi_2-\xi_1)^2z_{20}-3\varepsilon_2(\xi_2-\xi_1)z_{20}^2=0\,.$$

Summing the last two relations (7) results $\xi_1=0$ and, replacing in the third relation (7), we deduce

$$\xi_2\left(\varepsilon_2\xi_2^2-3\varepsilon_2\xi_2z_{20}+3\varepsilon_2z_{20}^2+k_2\right)=0\,. \tag{8}$$

We immediately obtain a first solution, name it $\xi_2=0$.

In the case of the hard stiffness it follows $\varepsilon_2>0$. In this situation, the second order equation in (8) has the discriminate

$$\Delta=-3\varepsilon_2^2z_{20}^2-4\varepsilon_2k_2<0\,. \tag{9}$$

Therefore, the only equilibrium position is given by

$$\xi_1=\xi_2=\xi_3=\xi_4=0\,. \tag{10}$$

In the case of the soft stiffness we have $\varepsilon_2<0$. From relation (9) is easy to see that for $\varepsilon_2\in\left[-\frac{4}{3}\frac{k_2}{z_{20}},0\right)$ the discriminate can be positive and two new solutions can appear. Thus, in this situation, generally speaking, we have three equilibrium positions: one given by relation (10) and other two obtained from (10) replacing ξ_2 by the solution of the second order equation in (8).

3. STABILITY OF THE EQUILIBRIUM POSITION FOR HARD STIFFNESS

The system (6) can be written in the form

$$\frac{d\xi_1}{dt}=\xi_3\,;\;\frac{d\xi_2}{dt}=\xi_4\,;\;\frac{d\xi_3}{dt}=-\frac{k_2+k_1+3\varepsilon_2z_{20}^2}{m_1}\xi_1+\frac{k_2+3\varepsilon_2z_{20}^2}{m_1}+NL_1\,;$$
$$\frac{d\xi_4}{dt}=\frac{k_2+3\varepsilon_2z_{20}^2}{m_2}-\frac{k_2+3\varepsilon_2z_{20}^2}{m_2}+NL_2\,, \tag{11}$$

where NL stays for non-linear terms. Denoting:

$$a_{31}=-\frac{k_2+k_1+3\varepsilon_2z_{20}^2}{m_1}\,;\;a_{32}=\frac{k_2+3\varepsilon_2z_{20}^2}{m_1}\,;\;a_{41}=\frac{k_2+3\varepsilon_2z_{20}^2}{m_2}\,;\;a_{42}=-\frac{k_2+3\varepsilon_2z_{20}^2}{m_2}\,, \tag{12}$$

the characteristic equation reads

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ a_{31} & a_{32} & -\lambda & 0 \\ a_{41} & a_{42} & 0 & -\lambda \end{vmatrix} = 0 \quad (13)$$

or, equivalently,

$$\lambda^4 - \lambda^2 (a_{42} + a_{31}) + a_{31}a_{42} - a_{41}a_{32} = 0. \quad (14)$$

The equation (14) is a two-quadratic one, and the second order equation corresponding to it has the discriminate

$$\Delta = (a_{42} - a_{31})^2 + 4a_{41}a_{32}. \quad (15)$$

Recalling the expression (12), we observe that $a_{31} < 0$, $a_{32} > 0$, $a_{41} > 0$, $a_{42} < 0$, so the discriminate is strictly non-negative. In addition, we have

$$a_{42} + a_{31} < \sqrt{(a_{42} + a_{31})^2 - 4a_{41}a_{32}}, \quad (16)$$

therefore the roots of the two-quadratic equation attached to the characteristic equation are both real and negative. We deduce that the characteristic equation has four pure imaginary roots and the equilibrium is a simply stable one.

4. STABILITY OF THE EQUILIBRIUM POSITIONS FOR THE SOFT STIFFNESS

In this case, the characteristic equation reads

$$\lambda^4 - \lambda^2 (a_{42}^* + a_{31}^*) + a_{31}^* a_{42}^* - a_{41}^* a_{32}^* = 0, \quad (17)$$

where we denoted:

$$a_{31}^* = -\frac{k_2 + 3\varepsilon_2 (\xi_2^*)^2 - 6\varepsilon_2 \xi_2^* z_{20} + 3\varepsilon_2 z_{20}^2 + k_1}{m_1};$$

$$a_{32}^* = \frac{k_2 + 3\varepsilon_2 (\xi_2^*)^2 - 6\varepsilon_2 \xi_2^* z_{20} + 3\varepsilon_2 z_{20}^2}{m_1}; \quad a_{41}^* = \frac{k_2 + 3\varepsilon_2 (\xi_2^*)^2 - 6\varepsilon_2 \xi_2^* z_{20} + 3\varepsilon_2 z_{20}^2}{m_2}; \quad (18)$$

$$a_{42}^* = -\frac{k_2 + 3\varepsilon_2 (\xi_2^*)^2 - 6\varepsilon_2 \xi_2^* z_{20} + 3\varepsilon_2 z_{20}^2}{m_2}.$$

In expressions (18) we marked by ξ_2^* the value of ξ_2 for one of the three equilibrium positions in this case. The two-quadratic equation (17) has the solution

$$\lambda_{1,2,3,4} = \pm \sqrt{\frac{\left(a_{42} - a_{32} - \frac{k_1}{m_1}\right) \pm \sqrt{\left(a_{42} - a_{32} - \frac{k_1}{m_1}\right)^2 + 4\frac{k_1}{m_1} a_{42}}}{2}}. \quad (19)$$

It is easy to see that if $a_{42} - a_{32} - \frac{k_1}{m_1} > 0$ or if $a_{42} - a_{32} - \frac{k_1}{m_1} < 0$ but $a_{42} > 0$ there exists at least one value λ, which is positive, therefore the equilibrium is unstable.

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