REACTIONS IN ONE DEGREE OF FREEDOM MULTIBODY SISTEMS WITHOUT SOLVING THE MOTIONS EQUATIONS

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Abstract In the paper is presented a multibody system with one degree of freedom, the aim being to compute the reactions of a such system. First, a kinematical analysis presents the liaisons between the generalized coordinates. These results are used in the dynamical analysis of the whole mechanism. Finally, the motion equations are presented. A natural method to eliminate the liaison forces is presented, method that replaces the well known method of the inversion of the coefficients matrix with a simple matrix multiplication that lightens significantly the numerical approach of the determination of reactions that appear in one freedom degree mechanisms. After this operation the determination of exterior reactions is an elementary operation.

Keywords: Multibody systems, liaison forces, virtual mechanical work

1. **INTRODUCTION**

The dynamic analysis of a system composed from many rigid bodies being in constraint can be accomplished with one of the following methods:

a) The linear momentum and kinetic moment theorems;
b) Lagrange equations;
c) The method of Lagrange multipliers;
d) Hamilton equations;
e) Energy theorem;
f) Virtual mechanical work principle,

written in different variants. If only the motion equations and their computing are having in view then the methods b), c) and d) are advantageous since they do not contain the unknown constraint forces. If the computing of constraint forces is necessary (for instance to conduct a strength calculus of the mechanism components) then the methods a) and d) must be used. In the following we will analyse the case a) because case d) can be reduced to this.

The equations that will be obtained applying method a) represent a system of second degree differential equations, in which near the independent coordinates unknowns, appear the constraint forces unknowns. These appear in linear combinations. Due to the different nature of the both types of unknowns, problems appear in resolving the equations systems. Usually, it begins in the first stage by eliminating the unknown constraint forces and then we proceed to compute the obtained equations system. In this case, the evolution in time of the independent coordinates being determined, a linear system which offers the constraint forces is computed. The elimination is possible because the unknown constraint forces intervene linear in equations, so an inversion technique of the coefficients matrix can be used.

In the following, a natural method to eliminate the constraint forces is presented, method that replaces the inversion of the coefficients matrix with a simple matrix multiplication, which lightens significantly the numerical approach of such problems. After this operation the determination of exterior reactions is an elementary operation.

2. KINEMATICS

We summarize the known results to express the kinematical conditions. Since, to write the motion equations is convenient to use the coordinates systems situated in the center of mass, we will compute the velocity and acceleration of the center of mass for each element. We will note with $\{\dot{x}\}$ the vector which contains on the first three positions the linear velocity components of the center of mass and on the following three positions, the components of the angular velocity vector of the element *i*. We will note with $\{\ddot{x}\}$ the vector which contains, in order, the linear acceleration and angular acceleration components of the center of mass of element *i*. If *s* is the mobility degree of the system imposed by constraints, then the vectors $\{\dot{x}\}$, *i=1,2,.. .,n*, where *n* is the number of bodies in constraint, can be linear expressed as a function of *s* derivatives q_j, *i=1,2,.. .,s*. We noted with q_j the independent coordinates of the system. So, it could be write:

$$\{\dot{x}\} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{Bmatrix} = [A] \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_s \end{Bmatrix} = [A]\{\dot{q}\} \tag{1}$$

In the following, it will be note:

$$\{\dot{q}_2\} = [\, \dot{q}_1^2 \quad \dot{q}_1\dot{q}_2 \quad \dot{q}_1\dot{q}_3 \; \dots \; \dot{q}_s^2 \quad]^T \tag{2}$$

The vectors $\{\ddot{x}\}_i$ *i=1,2,.. .,n* can be expressed by a linear function of $\ddot{q}_j$ and a quadratic function of $\dot{q}_j$, *j=1,2,.. .,s*. We have:

$$\{\ddot{x}\} = [A]\{\ddot{q}\} + [B]\{\dot{q}_2\} \tag{3}$$

relation obtained through the derivative of expression (1).

3. **MOTION EQUATIONS FOR MULTIBODY SYSTEMS**

We note with m_i the mass of the element *i*, with $[J]_i$ the matrix of inertia moments computed in the center of mass, $\{Q\}_i^{ext}$ the equivalent force-couple system of the exterior forces reduced in the center of mass and with $\{Q\}_i^{leg}$ the equivalent force-couple system of the constraint forces reduced in the center of mass. The motion equations for the element *i* can be written in the center of mass of the body:

$$[M]_i\{\ddot{x}\}_i + [M']_i\{\dot{x}\}_i = \{Q\}_i^{ext} + \{Q\}_i^{leg}$$

For the entire system we can write concise:

$$[M]\{\ddot{x}\} + [M']\{\dot{x}\} = \{Q\}^{ext} + \{Q\}^{leg} \tag{4}$$

If we take equation (3) into account we can write down:

$$[M][A]\{\ddot{q}\}+([M][B]+[M'][A])\{\dot{q}_2\}=\{Q\}^{ext}+\{Q\}^{leg} \tag{5}$$

which is a system of differential equations in independent coordinates $\{q\}$. In the equations appears the equivalent force-couple system of the constraint forces also with components in relation to each local coordinates system, disposed in the center of mass. In general, if *n* is the number of bodies, *s* the mobility degree, appear *6n – s* constraint forces. If we note with F_i, *i=1,2,.. ., 6n – s* the constraint forces, we can write down:

$$\{Q\}^{leg}=[T]\{F\} \tag{6}$$

where matrix $[T]$ depends on the system geometry. If the number of constraint forces is grater than *6n – s* we deal with a non-determined static structure, for the determination of the constraint forces, the elastic properties of the elements must be used.

4. WORK OF THE LIAISON FORCES

Let us consider $\{v_C\}_i$ the center of mass velocity of the element *i* of the system; $\{R\}_i$ a constraint force and A a point in which $\{R\}_i$ acts (fig. 1). The velocity of point A will be:

$$\{v_A\}_i=\{v_C\}_i+[w]_i\{r_A\}_i \tag{7}$$

where $[w]_i$ is the angular velocity operator of element *i*. The elementary mechanical work of the force $\{R\}_i$ at a displacement compatible with the constraints, is:

$$\begin{aligned} \mathrm{dL}_i &= \{R\}_i^T\{v_A\}_i dt=\{R\}_i^T(\{v_C\}_i+[w]_i\ \{r_A\}_i\ )dt= \\ &=(\{v_C\}_i^T\{R\}_i+\{w\}_i^T[r_A]_i\{R\}_i)dt=\{\dot{x}\}_i^T\begin{Bmatrix}\{R\}_i\\ [r_A]_i\{R\}_i\end{Bmatrix} \end{aligned} \tag{8}$$

But expression (8) represents the equivalent force-couple system of the force $\{R\}_i$ in point C. If we take into account all constraint forces and compute the total elementary mechanical work of the constraint forces that acts upon the element, we obtain:

$$\mathrm{dL}_i^e=\{\dot{x}\}_i^T\begin{Bmatrix}\{R\}\\ \{M_A\}\end{Bmatrix}=\{\dot{x}\}_i^T\{Q\}_i^{leg} \tag{9}$$

where $\{Q\}_i^{leg}$ is the equivalent force-couple system of the constraint forces in the center of mass C. For the whole system we can write down:

$$dL=dL_{1e}+dL_{2e}+\ldots+dL_{ne}=\{\dot{x}\}_1^T\{Q\}_1^{leg}+\ldots+\{\dot{x}\}_n^T\{Q\}_n^{leg}=\{\dot{x}\}^T\{Q\}^{leg}\ .$$

But the mechanical work of the constraint forces is zero, for the entire system we obtain:

$$\{\dot{x}\}^T\{Q\}^{leg}=0 \tag{10}$$

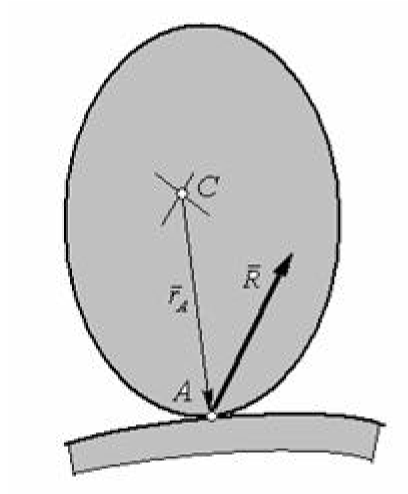

Fig. 1. A rigid body from the system, subjected to constraints

5. ELIMINATION OF CONSTRAINT FORCES

If we take into account the kinematic conditions (1), the expression (10) can be written:

$$\{\dot{q}\}^T [A]^T \{Q\}^{leg} = 0 \tag{11}$$

but the coordinates $\{q\}$ are independent, so that we obtain:

$$[A]^T \{Q\}^{leg} = 0 \tag{12}$$

This important result that can be proved in some general conditions will be used to eliminate the constraint forces from (5). So, pre-multiplying these relations with $[A]^T$, we obtain:

$$[A]^T [M][A]\{\ddot{q}\} + [A]^T ([M][B] + [M'][A])\{\dot{q}_2\} = [A]^T \{Q\}^{leg} + [A]^T \{Q\}^{ext} \tag{13}$$

and using (12), we can write down:

$$[A]^T [M][A]\{\ddot{q}\} + [A]^T ([M][B] + [M'][A])\{\dot{q}_2\} = [A]^T \{Q\}^{ext} \tag{14}$$

which represents a system of *s* second degree equations from which the constraint forces are missing. If we integrate this system, the constraint forces can be easily obtained. So, essentially, the elimination of constraint forces from the equations, reduces itself to the system multiplication with the matrix $[A]^T$.

6. AN EXAMPLE

We propose to determine the motion equations for a crank – connecting rod mechanism.

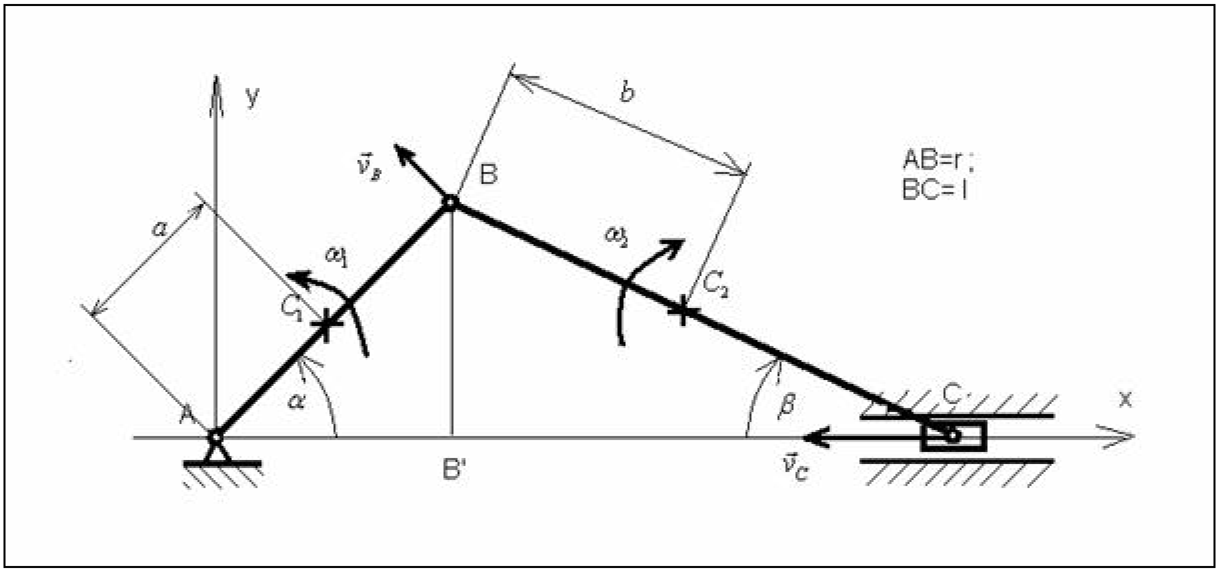

Fig. 2. The crank – connecting rod mechanism

Are given: *r* – the connecting rod radius; *l* – the crank length; *a* – the position of the connecting rod center of mass against point A; *b* – the position of the crank center of mass against point B.

Kinematical conditions (holonomic constraints):

$$x_{C1} = a\cos\alpha \ ; \quad y_{C1} = a\sin\alpha \ ; \quad x_{C2} = r\cos\alpha + b\cos\beta \ ;$$
$$y_{C2} = r\sin\alpha - b\sin\beta \ ; \quad x_C = r\cos\alpha + l\cos\beta \ ; \quad r\sin\alpha = l\sin\beta \ ; \qquad (15)$$

If the last relation is derived, we obtain:

$$r\dot{\alpha}\cos\alpha = l\dot{\beta}\sin\beta$$

or, if we take into account that:

$$\dot{\alpha} = \omega_1 \quad ; \quad \dot{\beta} = -\omega_2 \quad ; \quad \dot{\omega}_1 = \varepsilon_1 \quad ; \quad \dot{\omega}_2 = \varepsilon_2$$

it results:

$$r\omega_1\cos\alpha = -l\omega_2\cos\beta$$

or:

$$\omega_2 = -\frac{r\cos\alpha}{l\cos\beta}\omega_1 = \tau\omega_1 \ ; \text{ with } \tau = -\frac{r\cos\alpha}{l\cos\beta}$$

and:

$$r\varepsilon_1\cos\alpha - r\omega_1^2\sin\alpha = -l\varepsilon_2\cos\beta - l\omega_2^2\sin\beta$$

where:

$$\varepsilon_2 = -\frac{r\cos\alpha}{l\cos\beta}\varepsilon_1 + \left(\frac{r\sin\alpha}{l\cos\beta} - \tau^2\frac{\sin\beta}{\cos\beta}\right)\omega_1^2 = \tau\varepsilon_1 + \mu\omega_1^2 \text{ with } \mu = \frac{r\sin\alpha}{l\cos\beta} - \tau^2\frac{\sin\beta}{\cos\beta}$$

Therefore, the first relations, derived, will give:

$$\begin{Bmatrix} \dot{x}_{C1} \\ \dot{y}_{C1} \\ w_1 \\ \dot{x}_{C2} \\ \dot{y}_{C2} \\ w_2 \\ \dot{x}_C \end{Bmatrix} = \begin{Bmatrix} -aw_1 \sin a \\ aw_1 \cos a \\ 1 \\ -r\sin a + bt \sin b \\ r\cos a + bt \cos b \\ t \\ -r\sin a + lt \sin b \end{Bmatrix} w_1 = \{A_1\} w_1 \tag{16}$$

If we derive once more the kinematic conditions we obtain:

$$\begin{Bmatrix} \ddot{x}_{C1} \\ \ddot{y}_{C1} \\ e_1 \\ \ddot{x}_{C2} \\ \ddot{y}_{C2} \\ e_2 \\ \ddot{x}_C \end{Bmatrix} = \begin{Bmatrix} -a\sin a \\ a\cos a \\ 1 \\ -r\sin a + bt \sin b \\ r\cos a + bt \cos b \\ t \\ -r\sin a + lt \sin b \end{Bmatrix} e_1 + \begin{Bmatrix} -a\cos a \\ -a\sin a \\ 0 \\ -r\cos a - bt^2 \sin b + bu \sin b \\ -r\sin a + bt^2 \sin b + bu\cos b \\ u \\ -r\sin a + lt^2 \sin b + lu\cos b \end{Bmatrix} w_1^2 \tag{17}$$

or in a compact writing:

$$\{a\} = \{A_1\} e_1 + \{A_2\} w_1^2 \tag{17'}$$

Motion equations:

A possibility to obtain the motion equations is to apply the basic theorems for a rigid body. We consider the mechanism as being composed of three rigid bodies.

For the AB rod we obtain two equations from the linear momentum theorem and one considering the kinetic moment theorem:

$$m_1 \ddot{x}_{C1} = X_A + X_B$$

$$m_1 \ddot{y}_{C1} = Y_A + Y_B$$

$$J_{C1} e_1 = M_m + X_A \frac{r}{2}\sin a - Y_A \frac{r}{2}\cos a - X_B \frac{r}{2}\sin a + Y_B \frac{r}{2}\cos a$$

Fro the BC rod we obtain:

$$m_2 \ddot{x}_{C2} = -X_B + X_C$$

$$m_2 \ddot{y}_{C2} = -Y_B + Y_C$$

$$J_{C2} e_2 = X_C \frac{l}{2}\sin b + Y_C \frac{l}{2}\cos b + X_B \frac{l}{2}\sin b + Y_B \frac{l}{2}\cos b$$

Point C will have a rectilinear motion, so we can write:

$$m_3 \ddot{x}_C = F_r - X_C .$$

If we write all the grouped equations, we obtain:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_{C1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{C2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_{C1} \\ \ddot{y}_{C1} \\ e_1 \\ \ddot{x}_{C2} \\ \ddot{y}_{C2} \\ e_2 \\ \ddot{x}_C \end{Bmatrix} = \begin{Bmatrix} X_A + X_B \\ Y_A + Y_B \\ M_m + X_A a\sin a - Y_A a\cos a - X_B (r-a)\sin a + Y_B (r-a)\cos a \\ -X_B + X_C \\ -Y_B + Y_C \\ X_C (l-b)\sin b + Y_C (l-b)\cos b + X_B b\sin b + Y_B b\cos b \\ F_r - X_C \end{Bmatrix} \tag{18}$$

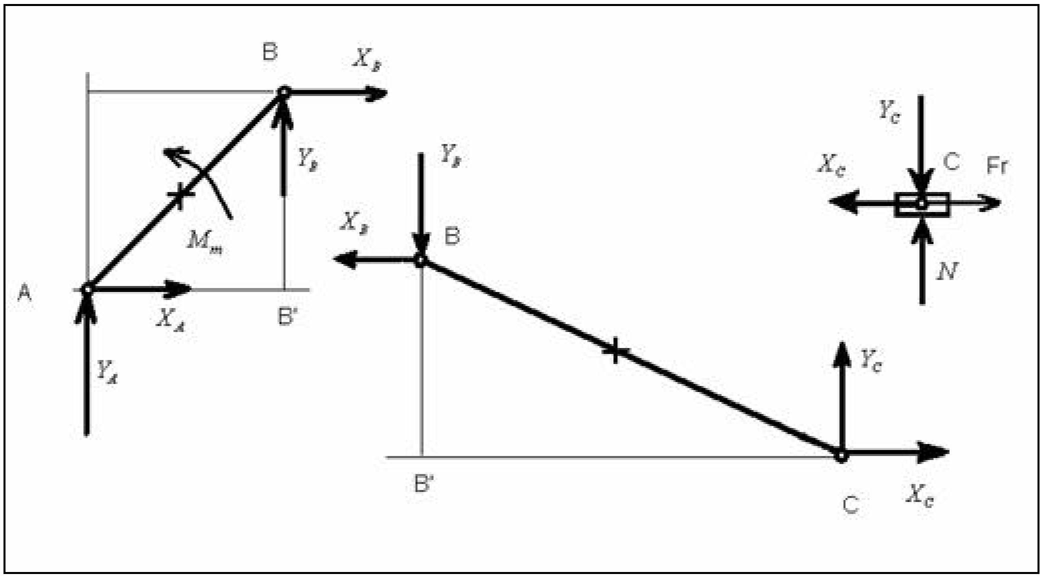

Fig. 3. The system separation in component parts

or, if we take into account the kinematic conditions:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_{C1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{C2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_3 \end{bmatrix} \left(\begin{Bmatrix} -a\sin a \\ a\cos a \\ 1 \\ -r\sin a + bt\sin b \\ r\cos a + bt\cos b \\ t \\ -r\sin a + lt\sin b \end{Bmatrix} e_1 + \begin{Bmatrix} -a\cos a \\ -a\sin a \\ 0 \\ -r\cos a - bt^2\sin b + bu\sin b \\ -r\sin a + bt^2\sin b + bu\cos b \\ u \\ -r\sin a + lt^2\sin b + lu\cos b \end{Bmatrix} w_1^2 \right) =$$

$$= \begin{Bmatrix} 0 \\ 0 \\ M_m \\ 0 \\ 0 \\ 0 \\ F_r \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ a\sin a & -a\cos a & -(r-a)\sin a & (r-a)\cos a & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & b\sin b & b\cos b & (l-b)\sin b & (l-b)\cos b \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} X_A \\ Y_A \\ X_B \\ Y_B \\ X_C \\ Y_C \end{Bmatrix} \qquad (19)$$

If we consider the previous notations we can write under grouped form:

$$[m](\{A_1\}e_1 + \{A_2\}w_1^2) = \{Q^{ext}\} + \{Q^{leg}\} = \{Q^{ext}\} + [N]\{R\} \qquad (19')$$

where the notations are obvious.

Mechanical work of the constraint forces:

The mechanical work of the constraint forces can be written as:

$$dL = \{d\,\Delta\}^T\{Q\}^{leg} = \{\dot{\Delta}\}^T\{Q\}^{leg}dt = \{\dot{q}\}^T[A_1]^T\{Q\}^{leg}dt = w_1\{A_1\}\{Q\}^{leg}dt$$

But we have the relation:

$$\{A_1\}^T\{Q\}^{leg}=0 \text{ so } dL=0 \ .$$

The vector $\{Q^{leg}\}$ represents the equivalent force-couple system of the generalized constraint forces according to the considered generalized coordinates. It results the motion equation:

$$J(a)\ddot{a}+J'(a)\dot{a}^2=M(a) \quad (20)$$

where the notations has been used:

$$J(a)=\{A_1\}^T[m]\{A_1\}=[m_1a^2+J_{C1}+m_2r^2+m_2b^2t^2+$$
$$+2m_2rbt\cos(a+b)+J_{C2}t^2+m_3(-r\sin a+lt\sin b)^2] \quad ;$$
$$J'(a)=\{A_1\}^T[m]\{A_2\}=$$
$$=[m_2(rbt^2\cos(a-b)+rbu\cos(a+b)-rbt\sin(a+b)-b^2t^3\cos 2b+$$
$$+b^2tu+J_{C2}tu+m_3(-r\sin a+lt\sin b)(-r\sin a+lt^2\sin b+lu\cos b)]$$
$$M(a)=M_m+F_r(-r\sin a+lt\sin b) \ ,$$

We obtain the motion equation under the form:

$$\frac{d}{dt}\left(\frac{J(a)\dot{a}^2}{2}\right)=M(a)\dot{a} \quad (21)$$

It results:

$$\frac{J(a)\dot{a}^2}{2}=\int M(a)da \ . \quad (22)$$

$$\dot{a}^2=\frac{2}{J(a)}\int_0^a M(q)dq \ ; \quad (23)$$

and:

$$\ddot{a}=\frac{1}{J(a)}\left(M(a)-J'(a)\frac{2}{J(a)}\int_0^a M(q)dq\right) \quad (24)$$

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