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FOURIER'S SERIES IN BIPOLAR CO-ORDINATES CORRESPONDING TO POLYNOMIAL CARTESIAN ELASTIC POTENTIAL

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Abstract. The Fourier's series in bipolar co-ordinate corresponding to polynomial Cartesian elastic potential are obtained. For the simplest forms of elastic potential stress patterns for elastic plane with two identical circular holes are shown.

1. INTRODUCTION

The aim of the paper is to obtain the expressions of Fourier expansion coefficients in bipolar co-ordinates for polynomial Cartesian potentials. The results are helpful in solving boundary problems in the elastic plane with two circular holes.

2. THEORETICAL CONSIDERATIONS

Problems from plane elastostatics can be solved either by using the Kolosov-Muskhelishvili, [1], equations from the complex potentials method, or by using Airy's stress function. The first method is based on complex variable functions theory and assumes finding two analytical functions used in obtaining stresses and strains. The second method presumes finding a function which, by differentiation, gives the stresses. For the case of the elastic plane with two circular holes, applying bipolar co-ordinates is suitable. The relation between the bipolar co-ordinates α and β and the Cartesian ones are:

$$x(\alpha,\beta) = a\frac{\sin(\beta)}{\cosh(\alpha)-\cos(\beta)},$$
$$y(\alpha,\beta) = a\frac{\sinh(\alpha)}{\cosh(\alpha)-\cos(\beta)}. \quad (1)$$

where a is a constant having length dimension. The next stages are assumed to be completed in solving a problem in plane elasticity, when two circular holes are made into an elastic plane:

- the Cartesian potential $U(x,y)$, characteristic for loading the compact plane at infinity, is expressed in bipolar co-ordinates

$$U(\alpha,\beta) = U[x(\alpha,\beta), y(\alpha,\beta)] \quad (2)$$

- - Jeffery (2), recommends that instead of using the potential $U(\alpha,\beta)$ for finding stresses in bipolar co-ordinates, the next potential should be used:

$$V(\alpha,\beta) = a\frac{U(\alpha,\beta)}{\cosh(\alpha)-\cos(\beta)}. \quad (3)$$

and presents the relations for stresses.

The global potential is created:

$$W(\alpha,\beta) = V(\alpha,\beta) + \Phi(\alpha,\beta) \quad (4)$$

where $\Phi(a,\beta)$ is an auxiliary potential, having a general form, also provided by Jeffery, which allows imposing boundary conditions on the contour of the holes; in bipolar co-ordinates , these contours have the following equations:

$$\alpha = \alpha_{01}, \\ \alpha = -\alpha_{02}. \quad (5)$$

- due to lack of domain limitations, conditions of stress regularity at infinity must be added to the boundary conditions:

$$\lim_{\alpha \to 0} \sigma_{\alpha\alpha} = 0, \\ \lim_{\alpha \to 0} \sigma_{\beta\beta} = 0, \\ \lim_{\alpha \to 0} \tau_{\alpha\beta} = 0, \quad (6)$$

As in bipolar co-ordinates, the point at infinity is characterized by:

$$\alpha = 0. \quad (7)$$

Trigonometric series expansion of the potential $V(\alpha,\beta)$ is needed in order to impose boundary conditions. For the usual loadings at infinity (mono-axial or hydrostatic traction or compression, shear, bending) the function $U(x,y)$ takes the form of a polynomial. Further more, for any other form of the function $U(x,y)$ an expansion of power series for variables x and y can be obtained. The stresses are obtained from the potential function by differentiation operations, which are linear. Accordingly, finding the power series expansion of the potential:

$$U(x,y) = x^m y^n, m \geq 0, n \geq 0, \quad (8)$$

is adequate. In bipolar co-ordinates, to the $U(x,y)$ function, the following function corresponds:

$$V(\alpha,\beta) = a \frac{\sin(\beta)^m \sinh(\alpha)^n}{[\cosh(\alpha) - \cos(\beta)]^{m+n-1}} \quad (9)$$

By denoting:

$$p = m + n - 1 \quad (10)$$

the Fourier's series with respect to β of the function

$$f(\beta)=\frac{1}{[\cosh(\alpha)-\cos(\beta)]^{p}} \tag{11}$$

The computation of coefficients of the expansion, a_k and b_k:

$$f(\alpha,\beta)=\frac{a}{2}+\sum_{k=1}^{\infty}[a_k\cos(k\beta)+b_k\sin(k\beta)] \tag{12}$$

The following relation is used:

$$\gamma_k=a_k+i\beta_k=\frac{1}{\pi}\int_0^{2\pi}f(\beta)e^{ik\beta} \tag{13}$$

With the following variable change

$$z=e^{i\beta},$$
$$dz=i\beta e^{i\beta}d\beta\Rightarrow d\beta=dz/ie^{i\beta}=dz/iz \tag{14}$$

the integral (13) takes the form:

$$\gamma_k=\frac{1}{\pi}\int_0^{2\pi}\frac{e^{ik\beta}}{[\cosh(\alpha)-\cos(\beta)]}d\beta=\frac{1}{\pi}\int_{|z|=1}\frac{z^k}{\left[\cosh(\alpha)-\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^p}\frac{dz}{iz}=$$
$$=\frac{2^p}{i\pi}\int_{|z|=1}\frac{z^{k+p-1}dz}{[z^2-2\cosh(\alpha)+1]^p}=\frac{2^p}{i\pi}\int_{|z|=1}\frac{z^{k+p-1}dz}{[z^2-2\cosh(\alpha)+1]^p}=2\pi i\sum Res[F(z)], \tag{15}$$

where by $\sum Res[F(z)]$ is meant the sum of the residues of the function

$$F(z)=\frac{z^{k+p-1}}{[z^2-2\cosh(\alpha)-1]^p} \tag{16}$$

in the inner points of the circle $|z|=1$. The function $F(z)$ can also be written in the form:

$$F(z)=\frac{2^p}{i\pi}\frac{z^{k+p-1}dz}{(z-e^{-a})^p(z-e^{\alpha})^p} \tag{17}$$

From equation 17, one can observe that the singularities of the function $F(z)$ are the points $z=e^{-\alpha}$ and $z=e^{\alpha}$ which are poles of *p* order. Among the two poles, only one is situated inside the $|z|=1$ circle, depending on $\alpha>0$ or $\alpha<0$ respectively.

The case $\alpha>0$ is considered next, the $\alpha<0$ being treated analogous.

The expression of the residue of a fraction in z_0 pole of *p* order, is, [3]:

$$\mathrm{Res}\,f(z_0) = \frac{[(z-z_0)^p f(z)]^{(p-1)}\,|_{z=z_0}}{(p-1)!} \tag{18}$$

Applying the relation (18) to the function $F(z)$ for $z_0 = e^{-\alpha}$ it results:

$$\mathrm{Res}\,F(e^{-\alpha}) = \left[\frac{d^{p-1}}{dz}\left(\frac{z^{k+p-1}}{(z-e^{a})^{p}}\right)\right]_{z=e^{-\alpha}} \tag{19}$$

For the calculus of the derivative, the following relation is applied:

$$(gh)^{(p-1)} = \sum_{s=1}^{n} \frac{(p-1)!}{s!(p-s-1)!} g^{(s)} h^{(p-1-s)} \tag{20}$$

where

$$g(z) = z^{k+p-1} \tag{21.a}$$

$$h(z) = (z-e^{\alpha})^{-p} \tag{21.b}$$

Carrying out the calculus leads to:

$$\frac{d^s}{dz^s} z^{k+p-1} = \frac{(k+p-1)!}{(k+p-s-1)!} z^{p+k-s-1} \tag{22.a}$$

$$\frac{d^{p-s-1}}{dz^{p-s-1}} = (-1)^{p-s-1} \frac{(2p-s-2)!}{(p-1)!} (z-e^{\alpha})^{-(2p-s-1)} \tag{22.b}$$

By substituting (22.a) and (22.b) in (20) and thereafter, applying in relation (19):

$$z = e^{-\alpha} \tag{23}$$

After a series of simplifying computations, the following form is obtained for γ_k:

$$a_k = \gamma_k = (-1)^{-p} \frac{2^{-(p-1)}}{(p-1)!} \sinh(\alpha)^{-2p} \cdot (k+p-1)!(e^{-\alpha})^{p+k} \cdot$$

$$\cdot \sum_{s=0}^{p-1} \frac{(2p-s-2)!}{s!(p-s-1)!} \frac{(e^{2\alpha}-1)^{s+1}}{(p+k-s-1)!}, \tag{24.a}$$

$$b_k = 0.$$

In a similar manner, for $\alpha < 0$, the following relation is obtained:

$$a_k = \gamma_k = (-1)^{-p} \frac{2^{-(p-1)}}{(p-1)!} \sinh(\alpha)^{-2p} \cdot (k+p-1)!(e^{\alpha})^{p+k} \cdot \sum_{s=0}^{p-1} \frac{(2p-s-2)!}{s!(p-s-1)!} \frac{(e^{-2\alpha}-1)^{s+1}}{(p+k-s-1)!}, \qquad (24.b)$$

$$b_k = 0$$

Considering the parity of the $f(\beta)$ function, it is predictable that:

$$b_k = 0, \qquad (25)$$

Another observation consists of different representation of the potential $V(\alpha,\beta)$ relative to the $x = 0$ line, on one side or the other.

The relations given by Rîjik and Gradstein, [8], are considered for the final result of the expansion of the $V(\alpha,\beta)$ function:

$$\sin(\beta)^{2q} = \frac{1}{2^{2q}} \left\{ \sum_{s=0}^{q-1} (-1)^{q-s} 2C_{2q}^{s} \cos[2(q-s)\beta] + C_{2q}^{q} \right\} \qquad (26.a)$$

$$\sin(\beta)^{2q-1} = \frac{1}{2^{2q}} \left\{ \sum_{s=0}^{q-1} (-1)^{q+s-1} 2C_{2q-1}^{s} \sin[2q-2s-1)\beta] \right\} \qquad (26.b)$$

where C_n^k represents binomial coefficient. As a_k are given by relations (24.a) and (24.b), the equation (9) takes the form:

$$V(\alpha,\beta) = \frac{\sin(\beta)^{2q} \sinh(\alpha)^n}{[\cosh(\alpha) - \cos(\beta)]^p} = \frac{1}{2^{2q}} \left\{ \sum_{s=0}^{q-1} (-1)^{q-s} 2C_{2q}^{s} \cos[2(q-s)\beta] + C_{2q}^{q} \right\} \cdot \left\{ \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\beta) \right\}. \qquad (27.a)$$

respectively:

$$V'(\alpha,\beta) = \frac{\sin(\beta)^{2q} \sinh(\alpha)^n}{[\cosh(\alpha) - \cos(\beta)]^p} = \frac{1}{2^{2q-2}} \left\{ \sum_{s=0}^{q-1} (-1)^{q+s-1} 2C_{2q-1}^{s} \sin[(2q-2s-1)\beta] \right\} \left\{ \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\beta) \right\} \qquad (27.b)$$

With the intention of imposing the boundary conditions, is required to obtain the expressions (27.a) and (27.b) in the form:

$$\frac{A_0}{2}+\sum_{k=1}^{\infty}[A_k\cos(k\beta)+B_k\sin(k\beta)] \tag{28}$$

Accordingly, the products of trigonometric functions occurring in relations (27.a) and (27.b) are transformed into sums. Therefore, the following form is obtained:

$$\cos[2(q-s)\beta]\cos(k\beta)=\frac{\cos[(2q-2s+k)\beta]+\cos[(2q-2s-k)\beta]}{2} \tag{29.a}$$

$$\sin[(2q-2s-1)\beta]\cos(k\beta)=\frac{\sin[(2q+k-2s-1)]+\sin[(2q-k-2s-1)]}{2} \tag{29.b}$$

After transforming the products of trigonometric functions into sums, in relations (27.a) and (27.b) linear combinations of trigonometric functions: $\cos(k\beta)$, $\cos[(2q-2s+k)\beta]$, $\cos[(2q-2s-k)\beta]$, and $\sin[(2q+k-2s-1)]$, $\sin[(2q-k-2s-1)]$ respectively, occur.
In order to obtain a (28) form for these functions, the change of summation indices in infinite sums is required. Using the notation:

$$p=2(q-s), \tag{30}$$

The (27) expression can be obtained under the general form:

$$V(\alpha,\beta)=\sum_{k=0}^{\infty}P_k\cos[(k-p)\beta]+\sum_{k=0}^{\infty}P_k\cos[k\beta]+\sum_{k=0}^{\infty}P_k\cos[(k+p)\beta]. \tag{31}$$

After a series of calculus, the 31 expression can be written in the form of a trigonometric series, given by the relation:

$$V(\alpha,\beta)=P_0\cos(p\beta)+P_p+Q_0+\sum_{k=1}^{p-1}(P_{p+k}+Q_k+P_{p-k})\cos(k\beta)+$$
$$+\sum_{k=p}^{\infty}(R_{k-p}+Q_k+P_{p+k})\cos(k\beta). \tag{32}$$

In a similar manner, by denoting:

$$p'=2(q-s)-1 \tag{33}$$

the relation (27.b) can be written in the following form:

$$V'(\alpha,\beta)=\sum_{k=0}^{\infty}P'_k\sin[(k-p')\beta]+\sum_{k=0}^{\infty}Q'_k\sin[k\beta]+\sum_{k=0}^{\infty}R'_k\sin[(k+p')\beta]. \tag{34}$$

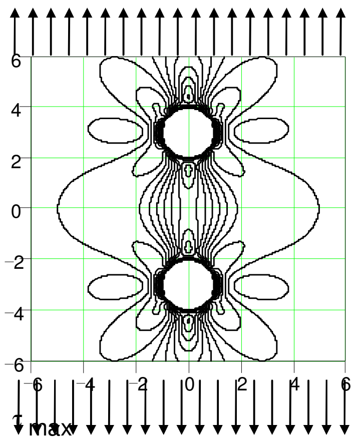
The relation (34) must be expressed as a trigonometric series of form (28), in order to impose the boundary conditions. After some intermediary calculus, for $V'(\alpha,\beta)$ potential the following form is obtained:

$$V'(\alpha,\beta) = -P'_0 \sin(p'\beta) + \sum_{k=1}^{p'-1} (Q'_k + P'_{p'+k} + P'_{p'-k}) \sin(k\beta) + \tag{35}$$

$$+ \sum_{k=p'}^{\infty} (P'_{p'+k} + Q'_k + R'_{k-p'}) \sin(k\beta)$$

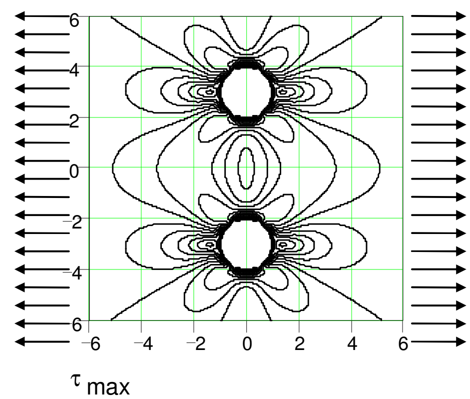
3. RESULTS

Some of the simple loadings of the elastic plane with two circular holes are presented in Figures 1-7, showing the contour lines of the principal stresses, obtained with the above deduced relations, in the present work. Figures 1 and 2 correspond to mono-axial traction, Figure 3 is obtained for hydrostatic traction, Figure 4 shows pure shear and Figures 5 and 6 present pure bending. The form of $U(x,y)$ potential is specified next to the figure.



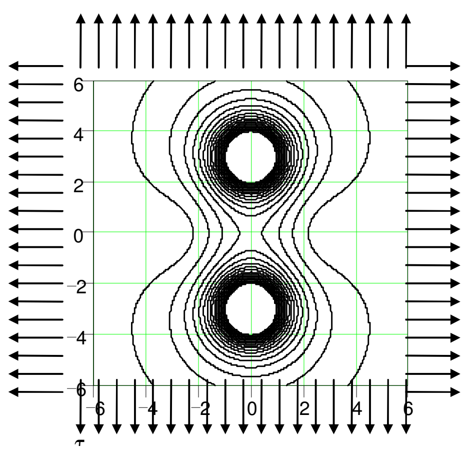
$U(x,y) = x^2/2$

Figure 1, [4]



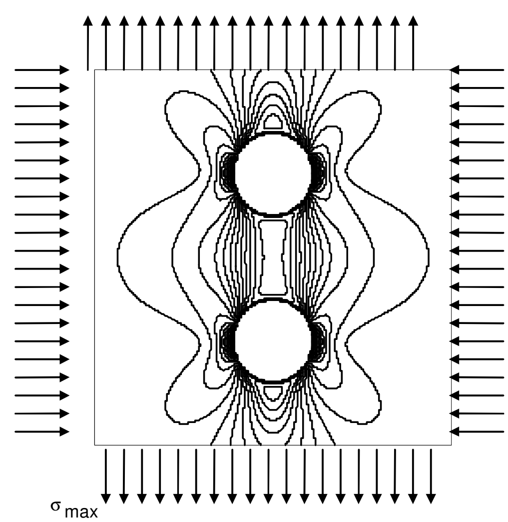
$U(x,y) = y^2/2$

Figure 2, [4]



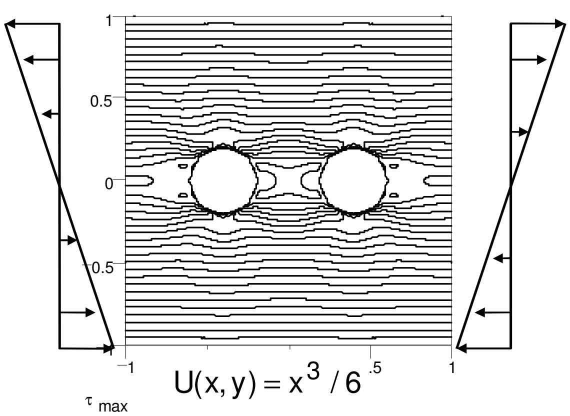
$U(x,y) = (x^2 + y^2)/2$

Figure 3, [4]



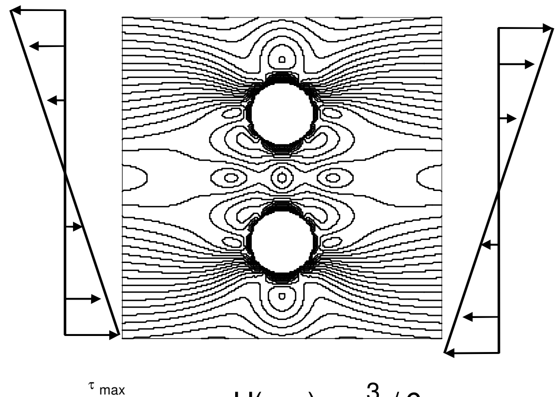
$U(x,y) = (x^2 - y^2)/2;$

Figure 4, [5]



$U(x,y) = x^3 / 6$

Figure 5, [6]



$U(x,y) = y^3 / 6$

Figure 6, [7]

4. CONCLUSIONS

The paper presents the calculus conducted for attaining the trigonometric series expansion of the elastic potential expressed in Cartesian co-ordinates as polynomials of two variables. The results are applied for finding the stress state in an elastic plane with two identical holes, acted at infinity by different loading.

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