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NEW APPLICATIONS OF THE PERFECT SEQUENCE TYPE FIBONACCI (ARRAY ϕ) ON GEOMETRICAL FIGURES IN PRODUCT DESIGN – 2

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The ratios generated by the sequence (ϕ) can be applied to geometric figures, regular or irregular, with multiple uses in product design. The paper presents new applications for triangles and rectangles.

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The simplest geometrical figure is the triangle. A triangle can be determined for which the angles are proportional with the (ϕ) sequence. For the triangle ABC with $\hat{A}$, $\hat{B}$ and $\hat{C}$.

$$\begin{cases} \frac{\hat{A}}{\hat{B}} = \frac{\hat{B}}{\hat{C}} = \phi \\ \hat{A} + \hat{B} + \hat{C} = 180^\circ \end{cases} \tag{1}$$

$$\hat{A} = \hat{B}\phi; \qquad \hat{B} = \hat{C}\phi; \qquad \text{therefore} \qquad \hat{A} = \hat{C}\phi^2 \tag{2}$$

Consequently

$$\hat{C}\phi^2 + \hat{C}\phi + \hat{C} = 180^\circ \tag{3}$$

It results that

$$\hat{C} = \frac{180^\circ}{\phi^2 + \phi + 1}, \tag{4}$$

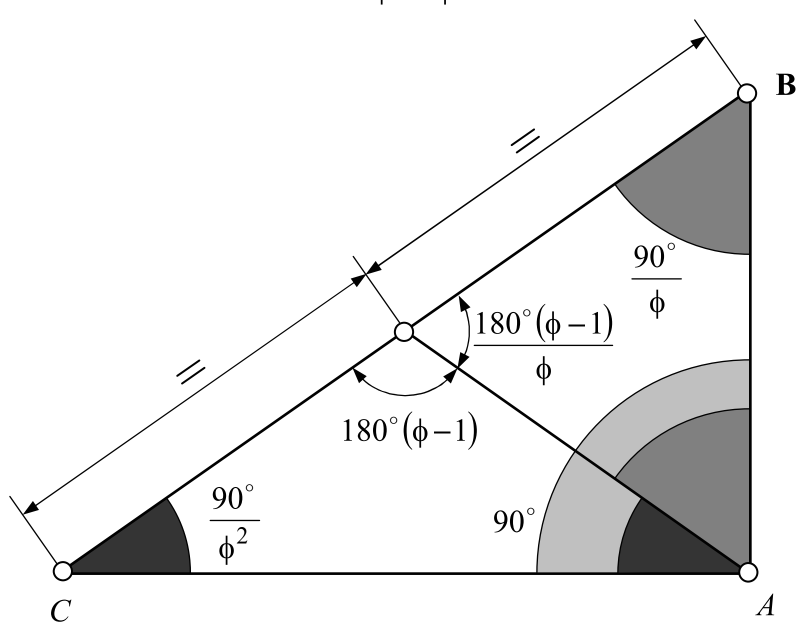

Fig. 1. The angles of the generated by the median from the right angle in a ϕ triangle

which leads to the following values:

$$\hat{C} = \frac{180^\circ}{\phi^2 + \phi + 1} = \frac{180^\circ}{2\phi^2} \cong 34{,}377^\circ \quad (5)$$

$$\hat{B} = \frac{180^\circ}{2\phi} \cong 55{,}623^\circ \text{ and} \quad (6)$$

$$\hat{A} = \frac{180^\circ}{2} = 90^\circ . \quad (7)$$

The conclusion is that the only triangle with angles proportional to the consecutive terms of the (ϕ) series is a right angled triangle. This triangle will be named „ϕ triangle".

In such a right angled triangle ϕ the median from the right angle A (fig.1) is splitting the 90° angle by the golden ratio ϕ. It is ascertained that the same median is splitting the 180° angle at the foot of the median line on the hypotenuse also by the golden ratio ϕ:

- an obtuse angle has

$$180^\circ - 2\frac{90^\circ}{\phi^2} = 180^\circ\left(1 - \frac{1}{\phi^2}\right) = 180^\circ\left(1 - \frac{1}{\phi}\right)\left(1 + \frac{1}{\phi}\right) = \\ = 180^\circ\left(1 - \frac{1}{\phi}\right)\cdot\phi = 180^\circ(\phi - 1) \cong 111{,}246^\circ; \quad (8)$$

- a keen angle has

$$180^\circ - 2\frac{90^\circ}{\phi} = 180^\circ\left(1 - \frac{1}{\phi}\right) = \frac{180^\circ(\phi - 1)}{\phi} \cong 68{,}754^\circ \quad (9)$$

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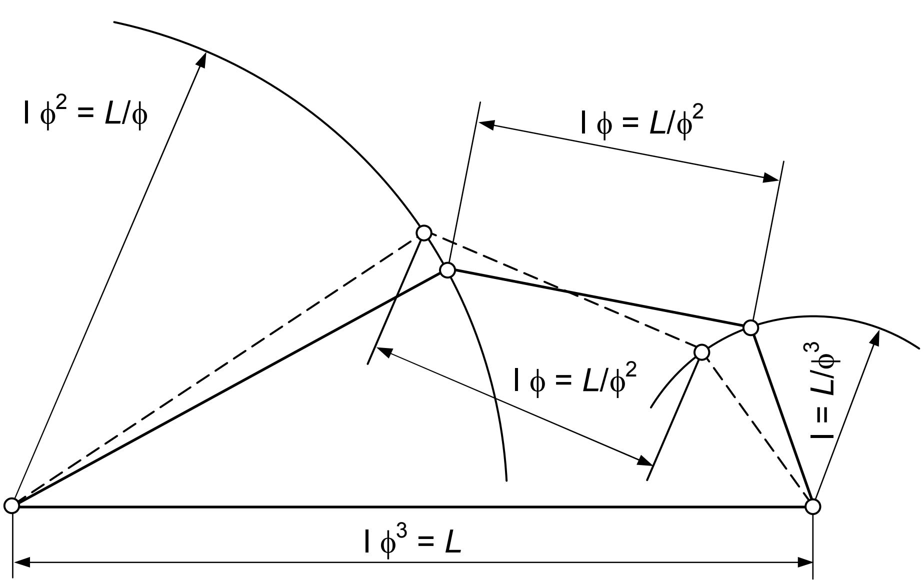

Fig. 2. Two variants (from an infinity) of quadrangles with the successive sides proportional with four consecutive elements of the ϕ or $\frac{1}{\phi}$ sequences.

Proceeding to the next plane figure, the quadrangle two types of ϕ quadrangles can be determined: with the sides by the (ϕ) series and with the angles by the (ϕ) series (as in the proportionality with 4 consecutive terms of the (ϕ) series).

It is obvious that there is an infinity of quadrangles which have the sides proportional with the terms 1, ϕ, ϕ^2 and ϕ^3 of the (ϕ) series (fig. 2).

The quadrangle ABCD (fig. 11-12) with the angles $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ proportional with the consecutive terms of the ϕ series is unique and is defined as following:

$$\begin{cases} \dfrac{\hat{A}}{\hat{B}} = \dfrac{\hat{B}}{\hat{C}} = \dfrac{\hat{C}}{\hat{D}} = \phi \\ \hat{A} + \hat{B} + \hat{C} + \hat{D} = 360^\circ \end{cases} \Rightarrow \begin{cases} \hat{C} = \hat{D}\phi \\ \hat{B} = \hat{D}\phi^2 \\ \hat{A} = \hat{D}\phi^3 \end{cases} \tag{10}$$

Then

$$\hat{D}\left(\phi^3 + \phi^2 + \phi + 1\right) = 360^\circ \tag{11}$$

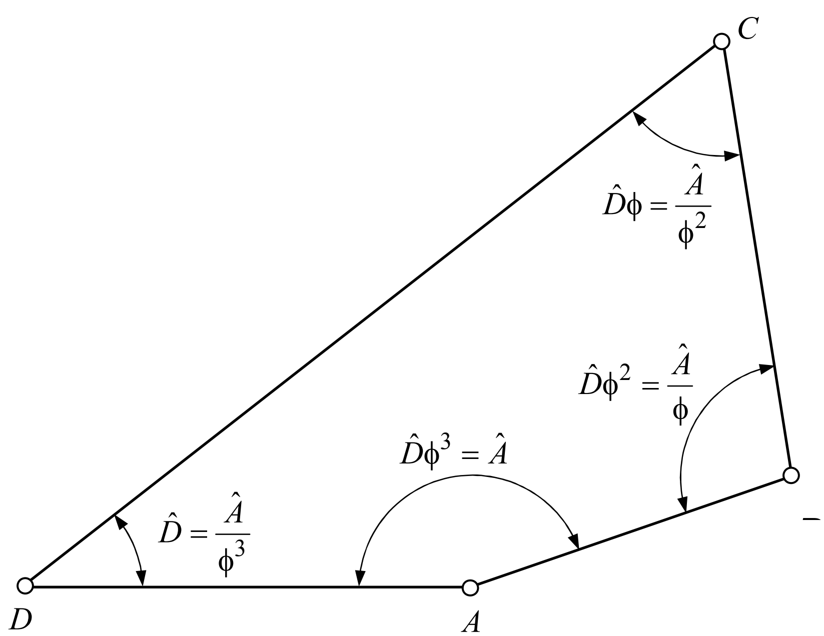

Fig. 3. The unique quadrangle with adjacent angles successively proportional with four elements of the $\frac{1}{\phi}$ ***series***

But

$$\phi^3 + \phi^2 + \phi + 1 = \phi^3 + \phi^2 + \phi^2 = \phi^3 + 2\phi^2 = \frac{\phi^4 - 1}{\phi - 1}, \tag{12}$$

because it can be observed that $1 + \phi + \phi^2 + \phi^3$ is the sum of the first four terms of a geometrical progression, with the first term $a_1 = 1$ and ration $q = \phi$. The following values are obtained:

$$\hat{D} = \frac{360^\circ}{\phi^3 + 2\phi^2} = \frac{360^\circ}{9{,}472136} \cong 38{,}0062112^\circ \cong 38{,}0^\circ ;$$
$$\hat{C} = \hat{D}\phi \cong 61{,}4953416^\circ \cong 61{,}5^\circ ; \qquad (13)$$
$$\hat{B} = \hat{D}\phi^2 \cong 99{,}5015528^\circ \cong 99{,}5^\circ ;$$
$$\hat{A} = \hat{D}\phi^3 \cong 160{,}9968944^\circ \cong 161{,}0^\circ .$$

In product design is a great emphasis placed on well proportioned rectangles, because this is one of the most used shape in products designed by men. The ϕ rectangle is the rectangle with the side ratio ϕ, respectively $\frac{1}{\phi}$. This type of rectangle is called ϕ rectangle or harmonic rectangle. The graphic construction of a ϕ rectangle (fig. 4) is obtained easily through the construction algorithm from fig 11-5 or fig. 11-7.

If quadrates are successively cut from a rectangle ϕ, a variety of figures are obtained which can be found as elements of the ϕ sequence (fig. 5, 6, 7, 8, 9), in the ratio between the found segments.

For example in fig. 5 the harmonic conjugated division was found $(A,B;T,T') = \phi$, presented in a different modality in fig. 5. Points *P*, *Q*, *A*, *B*, *R*, and so on are placed on a spiral ϕ^4, with the accumulation point O.

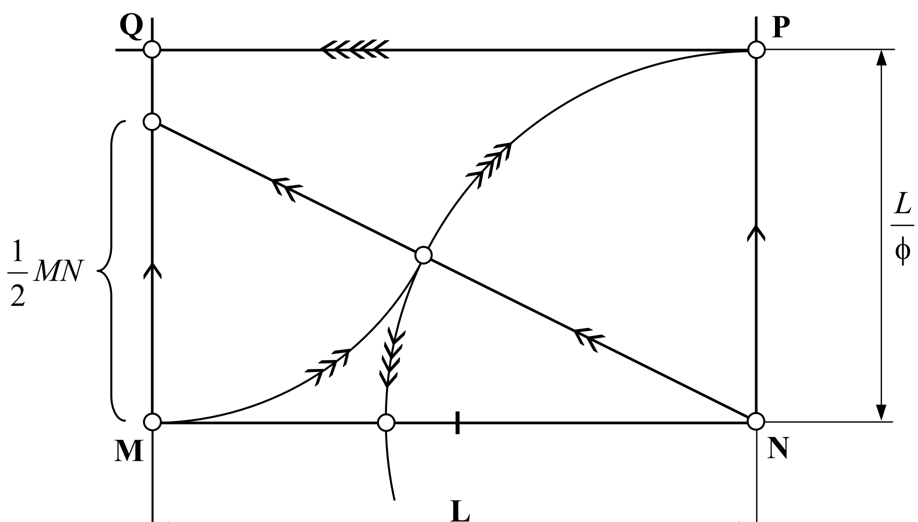

Fig. 4. Graphic algorithm for the construction of a ϕ ***rectangle***

In fig. 6, the homothety pole of the rectangles ϕ, the largest and the smallest, is point *O*, the intersection of the diagonals, and the homothety ratio is ϕ^3.

$$\frac{OP}{OQ} = \frac{OQ}{OA} = \frac{OA}{OB} = \frac{OB}{OR} = \phi ; \qquad \frac{OP}{OR} = \phi^4 \qquad (14)$$

$$AP \perp BQ$$
$$AP \cap BQ \equiv O \qquad (15)$$

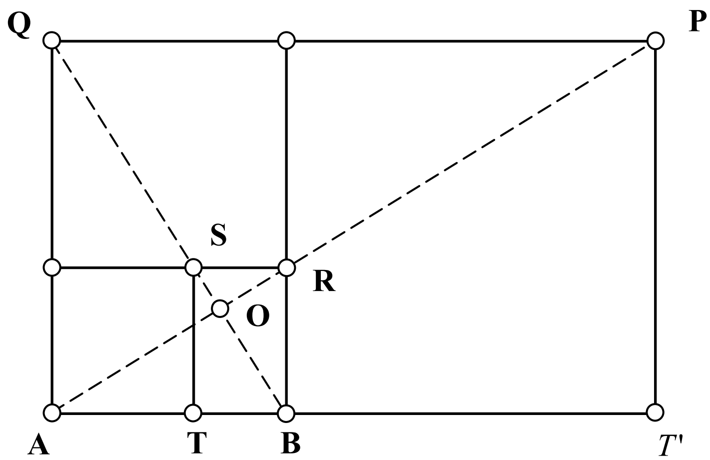

Fig. 5. Quarters and rectangles ϕ obtained successively

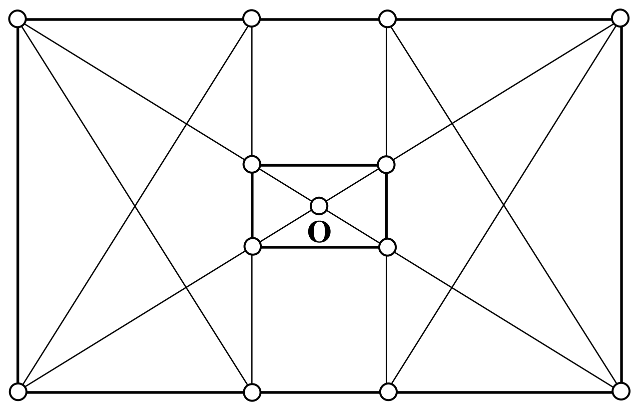

Fig. 6. Two rectangles ϕ homothetic of pole O

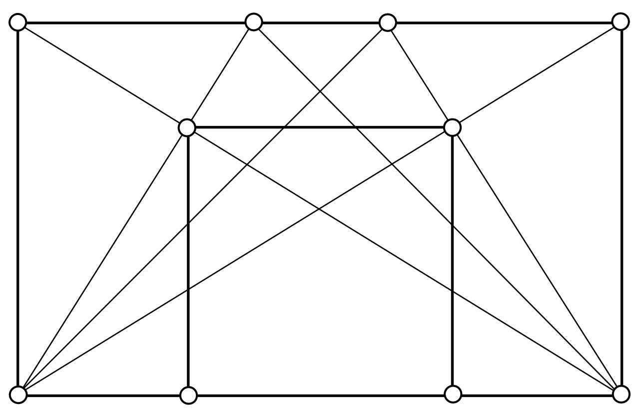

Fig. 7. Rectangle obtained from fig. 6

In fig. 8, all rectangles ϕ, as quarters in which the original figure is broken down, have for a homothety pole the point A.

In fig. 9, three homothetic rectangles (darkened with three shades of grey) of pole *P* were obtained.

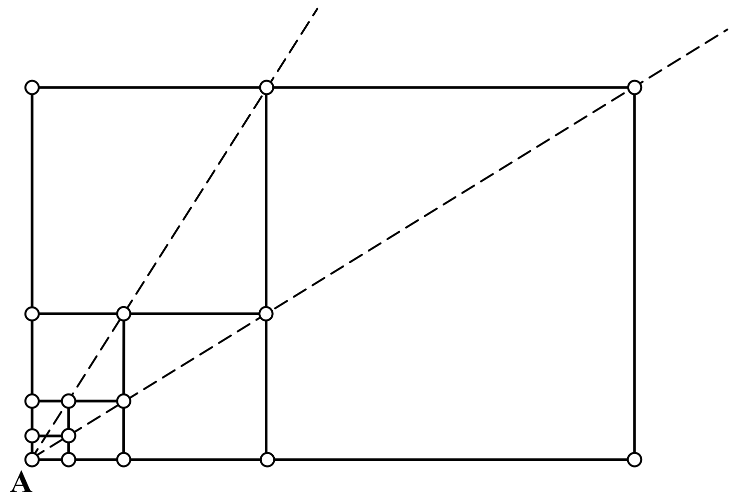

Fig. 8. Rectangles ϕ and quadrates having as a homothety pole the point A

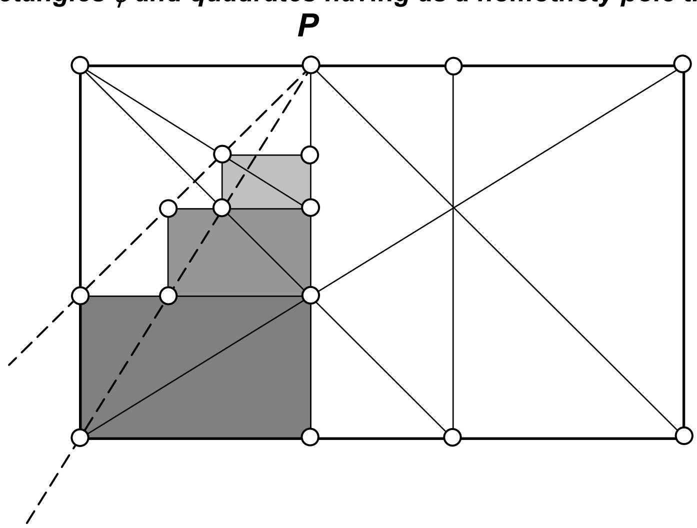

Fig. 9. The darkened rectangles are homothetic of point P

So far the triangle and the quadrangle ha been discussed. For the plane figures with a large number of sides, it is specified that the polygons that can be constructed with the ruler and compass have 3, 5, 17 sides. The following regular polygon can be constructed by ruler and compass and has 257 sides.

The number of sides of such a regular polygon is a prime number of the following form:

$$2^{2^k}+1 \text{ cu } k=0;\ 1;\ 2;\ 3;\ \ldots \tag{16}$$

Of course, all polygons with a number of sides acquired through the repeated doubling of 3, 5, 17, 257 etc. are constructed by ruler and compass.

The regular pentagon and decagon are polygons, which are „governed” by the golden ratio ϕ. Thus (fig. 10, fig. 11), in the regular convex pentagon, the ratio of diagonal and side is ϕ, and in the regular convex decagon (fig. 12), the ratio of circumscribed circle radius and side is the golden ratio ϕ.

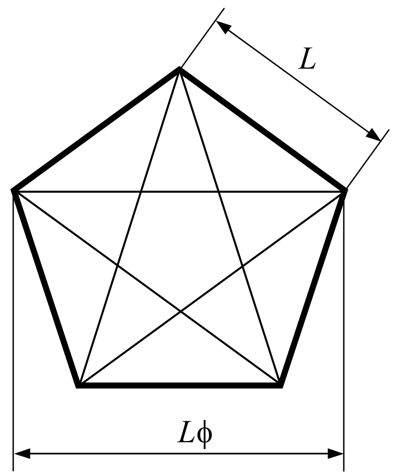

Fig. 10. Dimensional correlations with the golden ratio ϕ in the regular pentagon

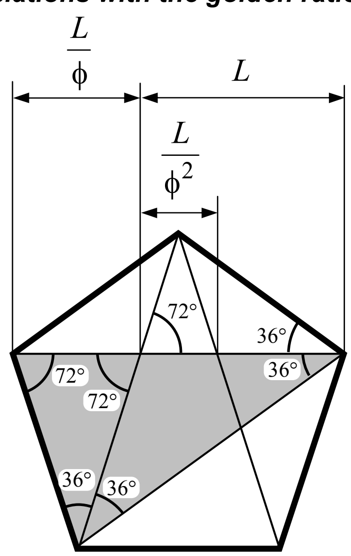

Fig. 11. Other dimensional correlations with the golden ratio ϕ in the regular pentagon

Differently put, if the radius of a circle is divided into medium and extrem ratio (construction can be render by ruler and compass), the larger part will represent the side of the decagon. Considered on a circle, this side determines an angle of 36° at the centre. By doubling this angle, a 72° angle is obtained at the centre, which correspondes to the side of a regular convex pentagon.

Once the principle and techniques to determine the harmonious proportions of angles, segments and plane figure are known and appropriated, you can easily progress to objects in space (volumes) built based on: parallelepipeds, prisms, pyramids, cones, frustums, conchoids or on any body obtained by the rotation of a plane surface around an axis or a point.

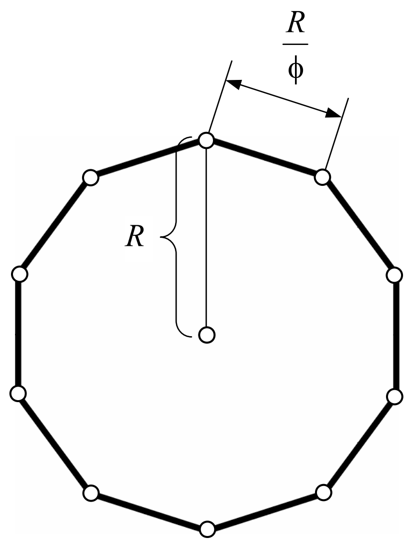

Fig. 12. Dimensional correlations with the golden ratio ϕ in the regular decagon

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