ANNALS of the ORADEA UNIVERSITY.
Fascicle of Management and Technological Engineering, Volume VI (XVI), 2007

ON THE LINEARIZATION TECHNIQUES FOR NONLINEAR DYNAMICAL SYSTEMS

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Keywords: nonlinear system, linearization, optimization, vibrations

Abstract. In this paper we present a new method for linearization of nonlinearly systems. The method can be used for nonlinear single or multi-degree-of-freedom-systems (SDOF or MDOF) and also for conservative and non-conservative systems, and for free and steady state vibration problems.

1. INTRODUCTION

Things that exist and surround us in the real world are exceptionally nonlinear, and the use of linear models is always a source of approximations. Linearization is(IS) often justified considering that:

- The study of linear system is far approachable than that of nonlinear systems and can be performed by using a set of second-order linear differential equation.
- The linearization of equations of motion allows the obtaining of interesting information on the behavior of the nonlinear systems. The behavior of the systems about an equilibrium position allows assessment of the stability of the equilibrium.

2. LINIARIZATION BY OPTIMIZATION METHOD

Consider the vibration of a nonlinear multi-degree-of freedom system (MDFO)

$$[M]\{\ddot{x}(t)\}+[C]\{\dot{x}(t)\}+[K]\{x(t)\}+[G(\{x\},\{\dot{x}\})]=\{F(t)\} \tag{1}$$

The linearization process by optimization method is obtained in next steps:

1^0.The first step is to find the exact solution, if available, or else one can use numerical techniques (such as Runge-Kutta method, θ-Wilson method or Newmark-β method) to obtain the time response $x_{ij}^{num}(t)$, $i=1,2,..,N;\ j=1,2,..,n$ where the indices i and j stand for the number of points selected from the time period T and the degrees of freedom in the system, respectively.

2^0. In the next step, one assumes an equivalent linear system

$$\begin{aligned}&[M]\{\ddot{x}(t)\}+[C+C_e]\{\dot{x}(t)\}+[K+K_e]\{x(t)\}=\\&[M]\{\ddot{x}(t)\}+[C_{lin}]\{\dot{x}(t)\}+[K_{lin}]\{x(t)\}=\{F(t)\}\end{aligned} \tag{2}$$

where $[K_{lin}]$ and $[C_{lin}]$ are stiffness and damping equivalent matrices, respectively.

The stiffness and the damping parameters are assumed, and the corresponding time history $x_{ij}^{eq}(t)$ is obtained from the modal analysis of the equivalent system of linear differential equations (2).

3^0. In the next step we introduce the optimization algorithm. For the general case one can define a vector of parameters $\{p\}$ whose components are stiffness and damping, $[C]$ and $[K]$ respectively, expressed as

$$\{p\} = \{k_1\, c_1\, k_2\, c_2 \,........ k_n\, c_n\}^T \tag{3}$$

Defining the error as

$$\delta_{ij} = x_{ij}^{num}(t) - x_{ij}^{eq}(t) \tag{4}$$

one defines an objective function as

$$V = \sum_{i=1}^{n} \sum_{i=1}^{n} \delta_{ij}^2 \tag{5}$$

Now one can state the constrained optimization problem as:

$$\begin{aligned} &\min V(k_1, c_1, k_2, c_2, ..., k_n, c_n) \\ &k_j^l \le k_j \le k_j^u \quad (j = 1,2,..,n) \\ &c_j^l \le c_j \le c_j^u \end{aligned} \tag{6}$$

Such types of problems can be solved using either direct search or gradient optimization methods [5], [10].

Thus the steps involved in this method for the *n*-degree-of-freedom are:

1. Assume a vector $\{p\}$ given by Eq. (2), within the upper and lower limits for the damping and stiffness parameters.
2. Obtain the time history of each of the degree-of-freedom within a time period using modal analysis of Eq. (2).
3. Calculate V given by Eq. (5).
4. Solve optimization problem given by Eq. (6).

For transient forced vibration response one can always calculate V using equations (4) and (5) and of course $x_{ij}^{eq}(t)$ must be obtained corresponding to an appropriate forcing functions.

3. APPLICATIONS AND COMPARATIVE STUDIES

3.1 Comparative studies

The response of the systems considered is found using:

1. a modified perturbation method [1];
2. weighted mean square method [4];
3. ultrasferical polynomials method [9];
4. optimization method.

The results can be compared to those obtained using the numerical solution.

3.1.1 Weighted mean square method versus optimization method

In the paper by Sinha and Srinivasan (1971) a linearization techniques is presented which yields an approximate period of oscillation T for the problem of free oscillations of conservative systems, governed by equation

$$\ddot{x} + f(x) = 0 \tag{7}$$

with the initial condition as $x(0) = x_0;\ \dot{x}(0) = 0$.
For the case of cubic nonlinearity (Duffing), the nonlinear function in equation (7) is given as $f(x) = \alpha\, x + \beta\, x^3$.
The approximation period of oscillation for the case of cubic non-linearity is found to be

$$T = 2\pi[\alpha + \frac{\lambda + 3}{\lambda + 5}\beta\, x_0^2]^{-\frac{1}{2}} \tag{8}$$

where $\lambda = 3$ is recommended to be used for the most accurate results.
This method of approximating the period of oscillation can be applied to free vibration of a conservative SDOF systems with cubic stiffness

$$m\ddot{x} + kx + k_1 x^3 = 0 \tag{9}$$

whit initial conditions $x(0) = x_0;\ \dot{x}(0) = 0$. With $\lambda = 3$, the approximate time period T becomes

$$T = 2\pi[\frac{k}{m} + \frac{3k_1}{4m} x_0^2]^{-\frac{1}{2}} \tag{10}$$

Using this period of motion obtained for the interval $(-x_0, x_0)$, a linearized equation of motion can be written as
The response is found using a modified equation

$$\ddot{x} + \omega_{lin}^2\ x = 0 \tag{11}$$

where $\omega_{lin} = T/2\pi$. The approximation solution for the equation (9) becomes

$$x(t) = x_0 \cos \omega_{lin} t \tag{12}$$

Using the optimization method of linearization, an approximate linearized function for equation (9) can be written as

$$\ddot{x} + \frac{k_{lin}}{m} x = 0 \tag{13}$$

and the solution becomes

$$x_{lin} = x_0 \cos(\sqrt{\frac{k_{lin}}{m}}\, t) \tag{14}$$

The linearized stiffness parameter k_{lin} was found so as to minimize the error between the exact solution and the approximate solution. Table 1 shows the calculated time period of the free vibration of the SDOF system and the value of the objective function. The calculated period is the same for all two methods, but the error *V* is the least for the optimization method.

Table 1: Free vibration of a SDOF system

Parameter	Weighted Mean Square Method	Optimization Method
T (sec)	0.4239	0.4239
V	0.00218	0.00064

3.1.2 Ultrasferical polynomials method versus optimization method

Consider a non-linear non-conservative system with a mixed type of non-linear damping and a cubic spring subject to step function excitation

$$m\ddot{x} + a(1+\dot{x}^2)\dot{x} + kx + bx^3 = F\eta(t) \tag{15}$$

where

$$\eta(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}. \tag{16}$$

An approximate solution of equation (15) are obtained by a generalized averaging technique based on ultrasferical polynomial extension [9], [4] and is given by

$$x(t) = C_0 - C(\tau)\cos(1+n)\tau \tag{17}$$

In equation (16) the non-dimensional time τ is given by $\tau = \sqrt{k/m}\,t$ and C_0 is the real root of equation $kx + bx^3 - F\eta = 0$.
The function $C(\tau)$ is given by

$$C(\tau) = \frac{C_0 e^{-\frac{a}{2}(1+C_0^2)\tau}}{[1 + \frac{C_0^2}{4(1+C_0^2)}(1 - e^{-a(1+C_0^2)\tau}]^{\frac{1}{2}}}. \tag{18}$$

In optimization method an approximate linear zed function for equation (15) is given by

$$m\,\ddot{x} + C_{lin}\,\dot{x} + K_{lin}\,x = F_0 \equiv F\eta(t);\, \dot{x}(0) = 0;\, x(0) = 0. \tag{19}$$

The general solution to the linear zed equation is [8].

$$x_{lin} = \frac{F_0}{K_{lin}\sqrt{1-\varsigma^2}}[\sqrt{1-\varsigma^2} - e^{-\varsigma\omega_n t}\cos(\omega t - \varphi)]. \tag{20}$$

Equation (15) was solved using the Runge-Kutta method. A multi-parameter optimization was used to find the linear zed damping constant C_{lin} and the linear zed spring constant

K_{lin} in equation (19), to minimize the error (4) between the Runge-Kutta solution and the approximate solution $x_{lin}(t)$.

Table 2 shows the error V calculated for a selected time period time period by the ultrasferical polynomials method and the optimization method respectively

Table 2: Forced vibration of a SDOF system

Parameter	Ultrasferical Polynomial Method	Optimization Method
T (sec)	6	6
V	0.001046	0.001032

4. CONCLUSIONS

The vibration response of non-linear single-degree-of-freedom (SDOF) or multie-degree-of-freedom (MDOF) whether conservative or non-conservative, free and forced, where formulated as optimization problems and solved. In these formulations, the optimization was based on the minimum error between the exact or numerical solution and an equivalent linear system solution. The results where compared with those obtained by other researchers. The results show tat this method is very versatile and relatively more accurate that the existing linearization methods.

5. REFERENCES

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