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ON COX-ROSS-RUBINSTEIN MODEL FOR EUROPEAN TYPE OPTIONS

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Abstract. This paper emphasizes the Cox-Ross-Rubinstein model used in financial mathematics for the technical evaluation of some options of european type. The model is a version in discrete time of the Black-Scholes model for derivatives.

Keywords. Portfolio, european type options CALL and PUT, derivatives, risky and nonrisky assets, random variable, expectation operator

1. INTRODUCTION

In recent years, the financial mathematics have witnessed an impressive development, due, in general, to the expansion of financial and capital markets and, in particular, to the diversification of the evaluation techniques used for the stock market products. The capital market, through its specific mechanism, concentrates the existent capitals and offers instruments and products able to fructify the investments and to cover the risk, being characterized by dynamism and innovation.

The problems aroused in matters as bank investments or stock market activities are related to the derivatives prices, hedging strategies or risk management, as well as portfolios policies and optimization.

The Cox-Ross-Rubinstein model is considered as an application in discrete time of the model built by Black and Scholes. It is used to determine the value of the derivatives and the level of risky assets that can be contained in an investor's portfolio.

2. PRELIMINARIES AND NOTATIONS

The european type options fall under the denomination for derivatives of securities and are represented by standardized contracts between a buyer and a seller, contracts that give the latest the right but not the obligation to sell, in what concerns PUT options, respectively to buy, for CALL options, a particular asset (as shares, bonds, goods), at a given price, namely the exercise price, at a certain ulterior date. This right is obtained by paying a prime to the seller of the derivative.

A CALL option is a financial instrument that offers the right to buy a certain asset at a recommended price and at a foreseen date. This option will be exerted at the maturity moment, denoted by T, only if the price of the share has a higher level then the exertion price.

A PUT option is a financial instrument that gives the opportunity to sell a certain asset at a suggested price and at a preestablished date. It is exerted only if the price of the share goes under the exertion price.

Let us consider that the given portfolio consists of two kind of assets, a nonrisky one and a risky asset, whose prices at the moment t, where $t = \overline{0,T}$, are denoted by S_t^0, respectively S_t. If one considers $S_0^0=1$, then following relation holds $S_t^0 = (1+i)^t$, where $i \in (-1,\infty)$ denotes the constant rate of return on a time unit.

We will denote by S_0 the initial market price. We suppose that the price of the risky asset on a unit of time verifies the relations

$$S_{t+1} = S_t \cdot (1+i_1) \text{ respectively } S_{t+1} = S_t \cdot (1+i_2),$$

where the unitary rates of return i_1 and i_2 verify the inequality $i_2 > i_1 > -1$.

Let us denote by $j = 1 + i$, $j_1 = 1 + i_1$ and $j_2 = 1 + i_2$, representing the actualization factors.

We distinguish two cases:

Case 1. If $j_1 \geq i$ the investor will borrow the amount of money S_0 in order to buy a unit of the risky asset. At the maturity moment T he will sell the risky asset and will return the loan. The positive profit will be denoted by $B_1 = S_T - S_0 j^T > 0$.

Case 2. If $j_2 \leq i$ the investor will sell short or without cover at the initial moment a unit of the risky asset at the price of S_0 and will invest the money in the nonrisky asset, which will provide at the maturity moment T the amount of $S_0 j^T$. After paying the buyer the sum of S_T, the positive benefit will be given by the difference $B_2 = S_0 j^T - S_T > 0$.

In what follows we will assume that

$$j_1 < j - 1 < j_2 \text{ and } j = \alpha \cdot j_1 + (1-\alpha) \cdot j_2, \text{ where } \alpha = \frac{i_2 - i}{i_1 - i}.$$

In general, to describe the financial market in a mathematical model, a finite space of probability (Ω, F, P) is considered, where Ω denotes a finite nonempty set, F the family of all subsets of Ω and $P(\{\omega\}) > 0,\ \forall\ \omega \in \Omega$. By $(F_t)_{t\in\{0,..,T\}}$ we denote the σ - algebra that represents the information available at the moment t.

As usual in the modelling of the financial market we will consider the elements of the space of probability as follows: $\Omega = \{j_1, j_2\}^T$, where a T-uple represents successive values of the variable denoted $(R_t)_{t\in\{0,\ldots,T\}}$ defined by the relation $S_t = R_t \cdot S_{t-1}$, where $t = \overline{0,T}$, F the family of all subsets of Ω and, as a particular case, $F_0 = \{\emptyset, \Omega\}$.

An option of european type with maturity T is defined by the level of its payoff, modelled as a nonnegative and F_T– measurable random variable.

Let us consider two european type options CALL and PUT whose values at moment t are denoted by $Call(t)$ respectively $Put(t)$. The maturity moment is T. The options are related to a stock market quotable asset. The values of the options depend on the financial asset on behalf of the exertion price denoted by s and which represents the amount of money due to buy (the CALL case) respectively to sell (the PUT case) the supporting asset.

Let s be a nonnegative and F_T– measurable random variable that denotes the options payoff. If the price of the supporting asset is considered $(S_t)_{t\in\{0,\ldots,T\}}$, then the values at maturity of the CALL options and PUT options will be given by

$$Call(T) = \max\{S_T - s, 0\} \text{, respectively } Put(T) = \max\{s - S_T, 0\}.$$

In a complete financial and capital market, there exists the possibility of an option to be swapped with a supporting asset. This is not always the case of an incomplete market where the transactions costs and the stochastic volatility must be taken into account.

3. THE MAIN RESULTS

This section presents some results on the value of a CALL option, a PUT option, as well as the amount of risky assets that can be contained in a portfolio at a given moment.

Theorem 3.1. *The value of a european CALL option is given by the relation*

$$Call(t) = \sum_{u=\overline{0,T-t}} C_{T-t}^{S} \cdot p^{u} \cdot q^{T-t-u} \cdot j^{T-t} \cdot \max\{\overline{S}_u - s, 0\},$$

where p and q are probabilities given as

$$p = P(R_1 = j_1) \text{ and } q = P(R_1 = j_2) \text{ such that } p + q = 1,$$

$$R_1 = \frac{S_1}{S_0} \text{ and } \overline{S}_u = S_t \cdot j_1{}^{u} \cdot j_2{}^{T-t-u}.$$

Proof. The returns variable denoted R_t defined by the relation $S_t = R_t \cdot S_{t-1}$, for all $t=\overline{0,T}$, satisfies the following equality

$$S_T = S_t \cdot \prod_{u=\overline{t+1,T}} R_u, \ \forall t = \overline{0,T}.$$

The returns are modelled as random variables, characterized by the expected value and a covariance value. The product $\prod_{u=\overline{t+1,T}} R_u$ is independent relative to F_t and S_t is F_t – measurable.

The value of the CALL option can be determined as an expected value in the next equalities

$$Call(t) = E\left[\max\left\{ j^{T-t}\left[S_t \cdot \prod_{u=\overline{t+1,T}} R_u - s\right], 0\right\} \mid F_t\right] = j^{T-t} E\left[\max\left\{\left[S_t \cdot \prod_{u=\overline{t+1,T}} R_u - s\right], 0\right\}\right].$$

By $E[Z]$ is denoted the expectation operator of an random variable Z.

The relation of the conclusion follows according to the distribution of the random variable $R = (R_t)_{t\in\{1,\dots,T\}}$. □

According to the following proposition, on behalf of the value of a CALL european option the value of a PUT european option can be also establish.

Proposition 3.2. *The connection between the values of the european options CALL and PUT and the price of the supporting asset at the moment t is given by the relation*

$$Call(t) - Put(t) = S_t - s \cdot j^{t-T}, \ \forall t = \overline{0,T}.$$

Proof. According to the definitions of the values of the european options, following relations hold

$$Call(t) \cdot j^{T-t} - Put(t) \cdot j^{T-t} = E\left[\max\{S_T - s, 0\} - \max\{s - S_T\} \mid F_t\right] =$$
$$= E\left[S_T - s \mid F_t\right] = E\left[S_t j^{T-t} - s \mid F_t\right].$$

As the actual price of the asset, denoted by $(\overline{S}_t)_{t\in\{0,\dots,T\}}$, is a martingale, we have

$$E[S_T \mid F_t] = E\left[\overline{S}_T \cdot j^{T} \mid F_t\right] = \overline{S}_t \cdot j^{T} = S_{t-1} \cdot j^{T-t} \cdot R_t.$$

The conclusion follows immediately. □

The selection of the contents of a portfolio means that the investor has to decide which number of assets and of shares he would like to possess.

The following result presents the proper amount of risky assets that can be contained in an investor's portfolio at a given moment *t*, in order to optimize his wealth at time *T*.

Theorem 3.3. *The amount of the risky supporting assets held by an investor in a portfolio at a certain moment t is given by the relation*

$$\nu_t = \frac{j^{T-t} \cdot (j_2 - j_1)^{-1}}{S_{t-1}} \cdot \sum_{u=\overline{0,T-t}} C_{T-t}^{u} \cdot p^{u} \cdot q^{T-t-u} \cdot j^{T-t} \cdot$$

$$\cdot \left[\max\left\{ S_{t-1} \cdot j_1^{u} \cdot j_2^{T-t-u+1} - s, 0 \right\} - \max\left\{ S_{t-1} \cdot j_1^{u+1} \cdot j_2^{T-t-u} - s, 0 \right\} \right],$$

where p and q are probabilities given by

$$p = P\left(\frac{S_1}{S_0} = j_1\right) \text{ and } q = 1 - p = P\left(\frac{S_1}{S_0} = j_2\right).$$

Proof. The value of the CALL option can also be given by means of the amount of the supporting assets, as in the following relation

$$Call(t) = \mu_t^{0} \cdot j^{t} + \nu_t \cdot S_{t-1} \cdot R_t,$$

where μ_t^0 is the quantity of the nonrisky supporting assets and ν_t the number of the risky supporting assets of a portfolio at a given moment *t*. Both μ_t^0 and ν_t are F_{t-1} – measurable and are depending on S_u, where $u \in \{1, 2, \ldots, t-1\}$.

We also have that

$$S_t = S_{t-1} \cdot j_1 \text{ and } S_t = S_{t-1} \cdot j_2, \text{ respectively } R_t = j_1 \text{ and } R_t = j_2,$$

where the unitary rates of return and the their present value versions satisfy the relation

$$i_2 > i_1 > -1, \text{ respectively } j_2 > j_1 > 0.$$

It follows, according to Theorem 3.1 that the value of the CALL option can be written as

$$Call(t) = \sum_{u=\overline{0,T-t}} C_{T-t}^{u} \cdot p^{u} \cdot q^{T-t-u} \cdot j^{T-t} \cdot \max\left\{ S_{t-1} \cdot j_k^{u} \cdot j_1^{u} \cdot j_2^{T-t-u} - s, 0 \right\} =$$

$$= \mu_t^{0} \cdot j^{t} + \nu_t \cdot S_{t-1} \cdot j_k, \text{ where } k \in \{1,2\}.$$

The relation of the conclusion follows by considering the difference of the previous relations, for $k = 2$, respectively $k = 1$. □

Remark. The structure of a portfolio assumes that the investor chooses the investments policies in various financial assets in order to optimize the utility of the consumption in a finite time lag *[0, T]*, respectively to optimize his wealth at the moment *T*.

4. BIBLIOGRAPHY

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