**ANNALS of the ORADEA UNIVERSITY.**
**Fascicle of Management and Technological Engineering, Volume VII (XVII), 2008**

# OPTIMUM SYNTHESIS OF PLANE QUADRANGULAR MECHANISMS USING A NUMERICAL METHOD

**Mircea LAZAR**
University of Pitesti, Faculty of Mechanics and Technology,
Department of Technology and Management
e-mail: mircea@lzr.ro

**Keywords:** trajectory, quadrilateral mechanism, optimum synthesis

**Abstract.** It is well-known that the synthesis of Cebâsev plane quadrangular mechanisms, for which the point of a connecting rod trajectory approximates a line segment, a circular arc or a curve is done on the basis of analytical, laborious methods, some of them being drawn up by Cebâsev himself. In this paper, it is presented a general method for the optimum synthesis of Cebâsev mechanisms by using the method of the smallest squares. The method is a numeric type and is based on the position function methods of numerical calculations, thus obtaining the optimum numeric synthesis of Cebâsev mechanisms. In order to apply this method, in the second part of the paper one will find the algorithm and the calculation program. Using several numerical examples, a range of such plane quadrangular mechanisms are determined.

## 1. Introduction

In specialty literature, the classical optimum synthesis of Cebâsev plane quadrangular mechanisms for which the point of a connecting rod trajectory approximates a line segment, a circular arc or a curve is done on the basis of analytical, laborious methods, some of them being drawn up by Cebâsev.

Usually, these analytical methods determine obtaining of non linear equations from which one finds out the quadrangular mechanism sizes; most of them are presented in various papers or scientific articles, manuals or specialty brochures.

Taking into account that some analytical methods are very laborious, a general method for synthesis of Cebâsev mechanisms is proposed in the paper. Mainly, the proposed method uses the smallest squares method, and then finally a numerical method is obtained; this is based on position functions and numerical calculations methods, thus getting the optimum numerical method synthesis of quadrangular mechanisms.

In order to make the demonstration of the proposed method, in the second part of the paper, one can find the algorithm and the calculation program needed for minimization of general functions; then for various values a series of Cebâsev mechanisms will be deducted, for which the trajectory of *M* point approximates circular arcs.

## 2. Establishing the necessary calculation relations

In order to establish the calculation relations, it is given the Cebâsev mechanism [2] in a XOY fixed system, like the one in Fig.1, in which a point of the mechanism describes $\Gamma$ curve. The following notations are used *OA* = a, *AB* = *BC* = *BM* = 1, *OC* = *d*, the angle of OA handle and BC arm with *Ox* axis is *f* , respectively *?,* and angle between *AB* and BM lines is $b$ .

The *AB* connecting rod is attached to local reference *Axy* system (fig.1); its position in relationship with *OXY* fixed reference system is given by the coordinates of *A* point (*a* cos f , *a* sin f ) and by ? angle which *Ax* axis forms with *OX* axis.

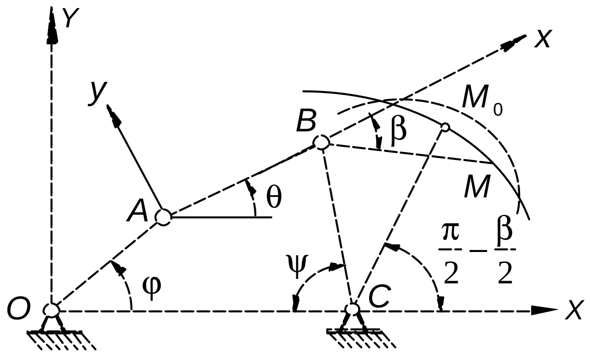


***Fig. 1. Cebâsev plane quadrangular mechanism***

Taking into account the notations in fig.1, the following coordinates of *M* point are found out in *Axy reference system*:

$$x = 1 + \cos ß;\ y = -\sin ß \tag{1}$$

but in *OXY* system, the *M* point coordinates are:

$$X = a\cos f + x\cos ? - y\sin ?$$
$$Y = a\sin f + x\sin ? + y\sin ? \tag{2}$$

From position equations:

$$a\cos f + a\sin ? - d = -a\cos ?;$$
$$a\sin f + \sin ? = \sin ? \tag{3}$$

by eliminating ? angle and marking cu $l = \frac{d}{a}$, we obtain the following relation:

$$q = 2arctg\frac{-B+\sqrt{A^2+B^2-C^2}}{C-A} \tag{4}$$

where:

$$A = 2a(\cos j - l);\ B = 2a\sin j;\ C = a^2(1+\lambda^2-2\lambda\cos\varphi) \tag{5}$$

## 3. General position function

We consider the plane quadrangular mechanism from fig.1, in which the trajectory described by *M* point (the curve drawn with continuous line) should approximate an arc of circle (the curve drawn with dotted line), with the equation:

$$a_1(X^2 + Y^2) + a_2X + a_3Y + 1 = 0 \tag{6}$$

If for f angle, $f_i$ positions are taken, where *i* = 1, 2,….., *N, for the M* point, the general coordinates $X_i$, $Y_i$ are obtained with the help of (2) relations. $e_i$ deviation of *M* point from the circular trajectory is obtained by introducing $X_i$, $Y_i$ general coordinates in (6) relation, from which the following expression results:

$$e_i = a_i(X_i^2 + Y_i^2) + a_2X_i + a_3Y_i + 1 \tag{7}$$

A good approximation of the deviation, in the respect of the smallest squares, is obtained when the sum:

$$S = \sum_{i=1}^{N} e_i^2 \tag{8}$$

is minimum, the value obtained when its derivatives in relationship with $a_1$, $a_2$, $a_3$ parameters is null. The following system is obtained:

$$\sum \varepsilon_i (X_i^2 + Y_i^2) = 0, \sum \varepsilon_i X_i = 0, \sum \varepsilon_i Y_i = 0 \tag{9}$$

totalization being done for $i = 1, \ldots, N$.

Under the conditions of (9)equations, the (8) sum becomes:

$$S = a_1 \sum \left(X_i^2 + Y_i^2\right) + a_2 \sum X_i + a_3 \sum Y_i + N \tag{10}$$

From the (9) and (10) relations, by eliminating $a_1$, $a_2$, $a_3$ parameters and using the notations:

$$\Delta = \begin{vmatrix} \sum \left(X_i^2 + Y_i^2\right)^2 & \sum X_i \left(X_i^2 + Y_i^2\right) & \sum Y_i \left(X_i^2 + Y_i^2\right) & \sum \left(X_i^2 + Y_i^2\right) \\ \sum X_i \left(X_i^2 + Y_i^2\right) & \sum X_i^2 & \sum X_i Y_i & \sum X_i \\ \sum Y_i \left(X_i^2 + Y_i^2\right) & \sum X_i Y_i & \sum Y_i^2 & \sum Y_i \\ \sum \left(X_i^2 + Y_i^2\right) & \sum X_i & \sum Y_i & N \end{vmatrix} \tag{11}$$

$$\delta = \begin{vmatrix} \sum \left(X_i^2 + Y_i^2\right)^2 & \sum X_i \left(X_i^2 + Y_i^2\right) & \sum Y_i \left(X_i^2 + Y_i^2\right) \\ \sum X_i \left(X_i^2 + Y_i^2\right) & \sum X_i^2 & \sum X_i Y_i \\ \sum Y_i \left(X_i^2 + Y_i^2\right) & \sum X_i Y_i & \sum Y_i^2 \end{vmatrix} \tag{12}$$

The final expression of the general position function is obtained:

$$S = \frac{\Delta}{\delta} \tag{13}$$

The established general position function (13) will contain a, ? and ß parameters. If synthesis refers only to mechanisms type connecting rod-balancing lever, then restrictions will appear:

$$0 < a < 1\ ;\ \ ? > 1;\ a\ (1 + ?) < 2 \tag{14}$$

When conditions for trajectory of *M* point should approximate a line, then in (8) and (10) sums the condition $a_1 = 0$ is imposed, and then the notations are used:

$$\Delta^* = \begin{vmatrix} \sum X_i^2 & \sum X_i Y_i & \sum X_i \\ \sum X_i Y_i & \sum Y_i^2 & \sum Y_i \\ \sum X_i & \sum Y_i & N \end{vmatrix};\ \ \delta^* = \begin{vmatrix} \sum X_i^2 & \sum X_i Y_i \\ \sum X_i Y_i & \sum Y_i^2 \end{vmatrix} \tag{15}$$

and the general position function is obtained:

$$S^* = \frac{\Delta^*}{\delta^*} \tag{16}$$

The required mechanisms are those for which *a*, ?, ß parameters make *S* and $S^*$sums in (13) and (16) relations to be with minimal values.

## 4. Establishing the trajectory

The (13) general position function depends on *a*, ? and ß parameters. After *a*, ? and ß parameters are found out, from (9) relations the $a_1$, $a_2$, $a_3$ coefficients are determined, using the relations:

$$a_i = -\frac{\delta_i}{\delta},\ i = 1, 2, 3; \tag{17}$$

where with $d_i$ was noted the determinants which are obtained from $d$ determinant, by replacing the column with the number *I* with column of $\left(\sum\left(X_i^2+Y_i^2\right),\sum X_i,\sum Y_i\right)^{\tau}$ form.

As $a_1$, $a_2$, $a_3$ coefficients are determined, these are replaced in (6) relation, after which the circle centre coordinates are found; and then the radius of the circle (the one which contains the circle arc proposed to be approximated) is found out using the relations:

$$X_C=-\frac{a_2}{2a_1};\quad Y_C=-\frac{a_3}{2a_1};\quad R=\sqrt{\frac{a_2^2+a_3^2}{4a_1^2}-\frac{a_3}{a_1}} \tag{18}$$

If the trajectory approximates a line segment, it results:

$$a^*{}_2=\frac{d_1^*}{d^*};\qquad a^*{}_3=\frac{d_2^*}{d^*} \tag{19}$$

where with $d_i^*$ were noted the determinants which are obtained from $d^*$ determinant by replacing the column number *I* with $\left(\sum X_i,\sum Y_i\right)^{\tau}$ column.

The equation of the line is:

$$a_2^*X+a_3^*Y+1=0 \tag{20}$$

In the following chapters, the algorithm and calculation program for minimizing the general position functions will be established and several Cebâsev mechanisms will be deducted for which the *M* point trajectory approximates circle arcs.

## 5. Properties of the connection rod curves for Cebâsev mechanisms

As it was shown at chapter 2, Cebâsev mechanisms (fig.1) are characterized by the fact that the lengths of *AB, BC, BM* segments are equal between them, i.e. they are considered equal to the unit, fact that implies determining a whole family of mechanisms.

If in the case of quadrangle in fig 1, the angle between AC and Ox axis is noted with $g$ (fig. 2) and considering the isosceles triangles *ABC* and *CBM,* it is found out that:

$$A\hat{C}M=\frac{\pi}{2}+\frac{\beta}{2}=\text{ct.} \tag{21}$$

which leads to the equality:

$$O\hat{C}A=M_0\hat{C}M=\gamma \tag{22}$$

where $M_0$ is the *M* point in the position in which $\varphi=0$.

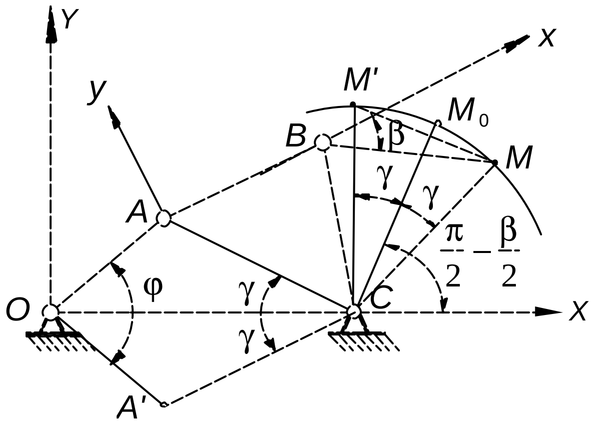


*Fig. 2. Kinematics scheme of Cebâsev type mechanisms*

Under these circumstances the following equality results:

$$A\hat{C}M = \frac{\pi}{2} + \frac{\beta}{2} = \text{ct.} \tag{23}$$

which leads to the equalities:

$$O\hat{C}A = M_0\hat{C}M = \gamma \tag{24}$$

where $M_0$ is the $M$ point in the position where the angle of *OA* handle is $j = 0$.

On the basis of the deducted relations one can notice that if the handle rotation with $j$ angle is considered, the *A* point arrives in *A'* position, and *M* point in *M'* point*, so that the latter is the symmetry of M* point towards $CM_0$ line. Thus, the following property is deducted: the *M* point trajectory is symmetrical to $CM_0$ line, and their symmetry is produced at the same time with the symmetry with *OA* handle towards *OC*. This property can be enunciated as following:

If we consider $M_0$ as the position of M point corresponding to angle f = 0 of OA handle, then the trajectory of M point is a curve symmetrical towards $CM_0$ axis, where $CM_0$ axis is inclined with angle $a = \frac{p}{2} - \frac{b}{2}$ towards OX axis.

The above property imposes the conclusion that if the *M* point trajectory approximates a circle arc, then this is symmetrical towards $CM_0$ line and the trajectory is described by rotating the *OA* handle from the $-j_1$ angle to $j_1$ angle, respectively by rotating from $j_2$ angle to $2p - j_2$ angle. The same thing happens also in the case of approximating the trajectory with a line segment.

Based on these properties and on the calculation relations established in chapter 2, the algorithm and calculation program can be established.

## 6. Calculation algorithm

For $j_1$ is considered an integral value in the 75° – 85° interval, after which the N position number is deducted:

$$N = 2\, j_1 + 1 \tag{25}$$

And indexation is done:

$$j_i = -j_1 + i - 1;\ \text{where}\ i = 1,2,.....N \tag{26}$$

Considering the notations *OA* = *a*; *OC* = *d*; $\lambda = \frac{d}{a}$, the (5) and (4) expressions are calculated for the general case, of the N positions, obtaining:

$$A_i = 2a(\cos j_i - l)\,;\, B_i = 2a \sin j_i\,, \tag{27}$$

$$q_i = 2arctg \frac{B_i + \sqrt{A_i^2 + B_i^2 - C_i^2}}{C_i - A_i} \tag{28}$$

The *x, y* coordinates of the *M* point in *Axy* system (fig. 1) are those established in (1) relation; the $X_i, Y_i$ coordinates of *M* point in *OXY* system:

$$X_i = a \cos f_i + x \cos ?_i - y \sin ?_i$$
$$Y_i = a \sin f_i + x \sin ?_i + y \sin ?_i \tag{29}$$

Further on, using (11) and (12) relations, the $\Delta$ and $d$ determinants are calculated and then the S general position function is obtained from (13) relation.

Taking into account the restrictions for obtaining the handle-balancing lever mechanisms (14), the following value intervals are chosen: $a \in [0,1;0,9]$, $l \in [1,1;11]$ and $b \in [-170^0;170^0]$.

For each set of values chosen in this way, the (13) general position function is calculated; its minimum values are kept, as well as the (*a,* ?, ß) values which make the minimum of *S* function.

For the values of *a,* ?, ß parameters which optimize the mechanism, the $a_i$ parameters are calculated, *i* = 1, 2, 3 using (17) relations.

$$a_i = -\frac{d_i}{d} \qquad (30)$$

where $d_i$ represent the determinants obtained from d determinant by replacing the column with *i* = 1, 2, 3 number with $\left(\sum\left(X_i^2 + Y_i^2\right), \sum X_i, \sum Y_i\right)^T$ column.

*The* $X_C, Y_C$ coordinates of the circle centre as well as the *R* radius is calculated using the following relations:

$$X_C = -\frac{a_2}{2a_1};\; Y_C = -\frac{a_3}{2a_1};\; R = \sqrt{X_C^2 + Y_C^2 - \frac{a_3}{2a_1}} \qquad (31)$$

If the trajectory is looked for within the $[j_2,\ 360° - j_2]$ interval, for the $j_2$ angle, it is considered a whole value within [100$^o$ – 110$^o$]; number *N* is calculated using the relation

$$N = 360 - 2j_2 + 1 \qquad (32)$$

and the following indexation is considered:

$$j_i = j_2 - 1 + i,\; i = 1, 2, ...N \qquad (33)$$

The further calculations are made on the basis of (6) – (15) relations.

## 7. Calculation program

For solving the optimum synthesis problem, a calculation program in Pascal programming language. The program contains, in order, the sequence of relations from point 2.

To cover the interval given by (13) relation three “For” cycles were used, in which (6)-(12) relations were cycled. The first *For* cycle is for the a parameter, the second is for the $\lambda$ parameter, and the last one is for the *b* parameter.

After each new value of a parameter, the optimum parameter results. These values are saved in a text type file; afterwards, based on this, tables with values for the mechanism sizes will be obtained.

The program allows also obtaining of script file based on which the successive trajectories of *M* point shall be obtained with the help of AutoCAD.

## 8. Numerical applications

***Case I.*** For $j_1 = 80°$, in which $a \in [0,2 - 0,8]$; $\lambda \in [1,1 - 11]$; $\beta \in [-100°;100°]$ it is considered that *a* and *l* parameters vary from decimal to decimal, and *ß* parameter varies from grade to grade. Based on the described calculation program, the results in table 1 were obtained with representations of trajectories and circle arcs in fig. 3.

***Case II***. For $j_2 = 110°$, in which $a \in [0{,}2 - 0{,}8]$; $\lambda \in [1{,}1 - 11]$; $\beta \in [-100°;100°]$, considering the same variation mode of *a,* ?, ß parameters, the results from table 2 were obtained and the representations of trajectories and circle arcs in fig. 4.

If the variation step of *a,* ?, ß parameters is chosen to be from hundredth to hundredth, then a greater number of optimized mechanisms can be determined.

*Table 1*

| Sum | a | l | b | d | R | $x_c$ | $y_c$ |
|---|---|---|---|---|---|---|---|
| 0.00001 | 0.200 | 4.400 | -50.000 | 0.880 | 2.357 | 1.03414 | -0.33055 |
| 0.00018 | 0.300 | 4.600 | 39.000 | 1.380 | 0.220 | 1.71693 | 0.95147 |
| 0.00121 | 0.400 | 3.600 | 39.000 | 1.440 | 0.279 | 1.77041 | 0.93305 |
| 0.00549 | 0.500 | 2.900 | 45.000 | 1.450 | 0.334 | 1.80917 | 0.86711 |
| 0.02069 | 0.600 | 2.300 | 61.000 | 1.380 | 0.381 | 1.79995 | 0.71294 |
| 0.06472 | 0.700 | 1.800 | -64.000 | 1.260 | 2.473 | 1.55242 | -0.46797 |
| 0.20719 | 0.800 | 1.400 | -52.000 | 1.120 | 2.238 | 1.26616 | -0.29967 |

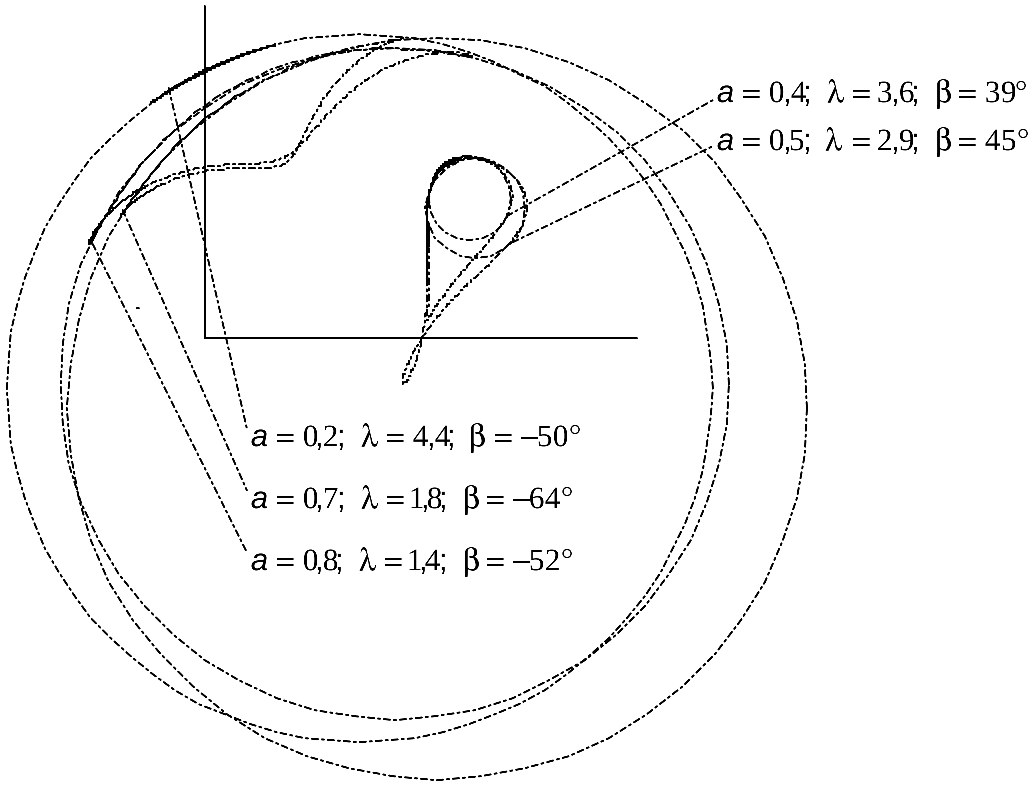


***Fig. 3. Trajectories obtained based on table 1.***

*Table 2*

| Sum | a | l | b | d | R | $x_c$ | $y_c$ |
|---|---|---|---|---|---|---|---|
| 0.00000 | 0.200 | 1.700 | 11.000 | 0.340 | 2.284 | 0.29989 | -0.41655 |
| 0.00005 | 0.300 | 3.200 | -8.000 | 0.960 | 2.135 | 0.69687 | 3.76292 |
| 0.00044 | 0.400 | 1.100 | -80.000 | 0.440 | 1.670 | 0.27201 | 0.20020 |
| 0.00131 | 0.500 | 1.100 | -80.000 | 0.550 | 1.752 | 0.40367 | 0.17439 |
| 0.00318 | 0.600 | 1.100 | 120.000 | 0.660 | 0.105 | 0.49835 | -0.09333 |
| 0.00578 | 0.700 | 1.100 | 120.000 | 0.770 | 0.186 | 0.43173 | -0.19530 |
| 0.01361 | 0.800 | 1.100 | 120.000 | 0.880 | 0.290 | 0.35713 | -0.30188 |

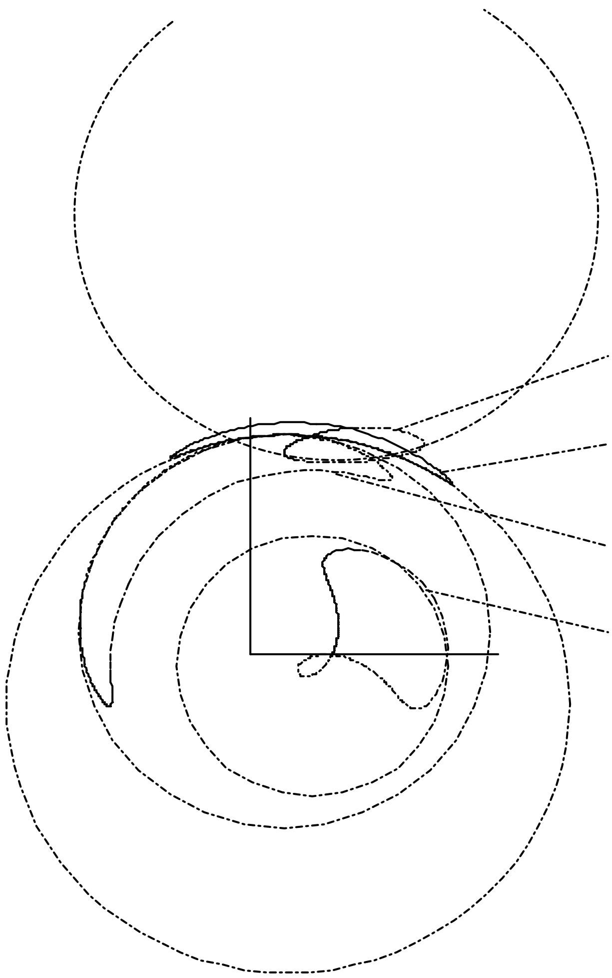


***Fig. 4. Trajectories obtained based on table 2.***

## 9. Conclusions

The method established and proposed as numerical method, the algorithm and the calculation program which were drawn up in this paper allow determination of a big enough number of Cebâsev type mechanisms. The mechanisms obtained through numerical applications are plane optimized quadrangular mechanisms, for which the *M* generator point describes trajectories that approximate circle arcs on a considered interval.

## References

[1] Artobolevski I. I., *Les mécanismes dans la tehnique moderne,* Tome 1-5, Editions MIR, Moscow, 1978.
[2] Lazar M ., Pandrea N., *Asupra curbelor de biela ale mecanismelor plane patrulatere articulate*, Sesiunea de comunicari stiintifice, Univ. „C. Brâncusi", Tg-Jiu, noiembrie, 2000.
[3] Lazar M., *Asupra sintezei optimale a mecanismelor Cebâsev*, Sesiunea comunicari stiint., Oradea, 2001.
[4] Pandrea N., Pelecudi Chr., *Determination of on obiective function for the best approximation on a given path on a sphere*, ASME Publication, 1974.
[5] Pandrea N., Lazar M., *Asupra sintezei optimale multipozitionale a mecanismelor plane*, Sesiunea de Comunicari Stiintifice, vol. „Mecanica aplicata'', pag 5 - 15, Sibiu, 1995.
[6] Pandrea N., Popa D., *Mecanisme. Teorie si aplicatii CAD*, Editura Tehnica, Bucuresti, 2000.
[7] Pelecudi Chr., Pandrea N., Draganoiu Gh., *Mathematical estension of Burmaster's theory in the optimum synthesis of plan linkages*, Proceedings of the Institute of Mathematics, Iasi, 1974.
[8] Sarkisyan Y., Gupta K., Roth B., *Kinematic Geometry Associated with the Least Squar Approximation of a Given Motion,* Transactions of the ASME, vol. 95, 1973.