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FLOQUET'S THEOREM – ITERATION SEQUENCE IN OSCILLATION THEORY

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Summary: By method of iteration sequences, introduced by the author in [5] and applied to the Hill's equation (1), first the general form of the solution is determined, if ϕ(x) is a periodic function, which is given with (11), where the "amplitudes" F(x) and G(x) are periodic with the same periods (and they can also be oscillating, even monotonic). Initially, a completely different and new method has been applied to prove the classical theorem on Hill's equation (although the other possibilities have also been pointed out). The authors think that the suggested iteration method is more adequate than the classical Hill's method, which has been repeated in literature for more than 150 years

Let there be the second order homogenous differential equation in normal form

$$y'' = \phi(x)y \tag{1}$$

where f is a continuous and periodic function in the interval [0,x], and is called Hill's equation.

The main question regarding this canonical form of the second order equation, if we have in mind that ϕ is a periodic function, concerns what the general form of the solution of this equation is, since the exact solution can be rarely found. The answer is provided in the form of historical Floquet's theorem ([1], [2]), which has become the basics for all latter monographs about Hill's equation ([3], [4]). First, let us observe the basic, but also the most important fact: since the solution of differential equations of the same type depends solely and exclusively on coefficients, in the case of (1) we naturally expect the periodic solutions. Since ϕ is periodic function according to our presumption, in that case there is a constant ω for every x, so it follows:

$$\phi(x+\omega) = \phi(x)$$

Does periodicity apply to all (or only to some) solutions?

$$y(x+\omega) = y(x)$$

If in (1), we put the value x+ω instead of x, there will be:

$$y''(x+\omega) = \phi(x+\omega)\cdot y(x+\omega) \tag{2}$$

Since for continuous coefficient ϕ, all solutions are continuous, and since the second derivative of the periodic function is also a periodic function

$$y''(x+\omega) = y''(x)$$

Therefore, in the last equation we have

$$y''(x) = \phi(x+\omega)\cdot\phi(x) \tag{3}$$

If we subtract (2) from (3) we get

$$[\phi(x+\omega) - \phi(x)]\cdot y(x) = 0$$

And since for non-trivial solution we have y(x)≠0, it remains:

$$\phi(x+\omega) - \phi(x) = 0$$

i.e.

$$\phi(\mathrm{x}+\omega)=\phi(\mathrm{x}) \qquad (4)$$

This implies

Theorem 1.

Hill's equation has periodic solutions, if the coefficient ϕ is a periodic function.

Primarily this means that canonical form (1) cannot have periodic solutions, if ϕ is not a periodic function. If it is periodic, we will see that periodic general solutions will be rare. This implies that the condition (4) is necessary, but not enough.

Therefore, apart from possible periodic solution, there is also another solution which does not have to be periodic. The main qualitative problem arising now involves the question of what the other linearly independent solution is.

By applying the method of iteration sequence, under the already stated presumption that ϕ(x) is also a continuous function, we get (5), i.e. the following general solution in the form of convergent integral:

$$\begin{aligned} y &= C_1y_1 + C_2y_2 = \\ & C_1[1+\int\int\varphi(\mathrm{x})\mathrm{dx}^2+\int\int\varphi(x)dx^2\int\int\varphi(\mathrm{x})\mathrm{dx}^2+ \\ &+\int\int\varphi(\mathrm{x})\mathrm{dx}^2\int\int\varphi(\mathrm{x})\mathrm{dx}^2\int\int\varphi(\mathrm{x})\mathrm{dx}^2+......] \\ &+C_2[x+\int\int x\varphi(\mathrm{x})\mathrm{dx}^2+\int\int\varphi(x)dx^2\int\int x\varphi(\mathrm{x})\mathrm{dx}^2+ \\ &+\int\int\varphi(\mathrm{x})\mathrm{dx}^2\int\int\varphi(\mathrm{x})\mathrm{dx}^2\int\int x\varphi(\mathrm{x})\mathrm{dx}^2+......] \end{aligned} \qquad (5)$$

This implies

Since ϕ(x) is a periodic function, it is the most natural thing to alter the sign from one interval to the other. (If ϕ(x) is periodic, but has the permanent sign, its solutions are permanently oscillating if
ϕ <0 or permanently monotonous if ϕ >0).

Therefore, let there be ϕ (x)>0 in the interval (x_k, x_{k+1}) and ϕ (x)<0 in the next interval (x_{k+1}, x_{k+2}). Let us consider first the interval where ϕ(x)>0. In this case all addends in upper integrals are positive, so the solutions are monotonous functions, as it has been shown earlier (5):

$$y_1 = ch_{\phi(\mathrm{x})}\mathrm{x} \approx \frac{1}{2}\left(e^{x\sqrt{\phi(\mathrm{x})}}+e^{-x\sqrt{\phi(\mathrm{x})}}\right) \qquad (6)$$

$$y_2 = sh_{\phi(\mathrm{x})}x \approx \frac{1}{\sqrt{\phi(\mathrm{x})}}sh(x\sqrt{\phi(\mathrm{x})} = \frac{1}{2\sqrt{\phi(\mathrm{x})}}\left(e^{x\sqrt{\phi(\mathrm{x})}}-e^{-x\sqrt{\phi(\mathrm{x})}}\right)$$

In the next interval let there be ϕ(x)<0. In this case the addends in the sequence (2) have alternating signs. Thus we get:

$$\begin{aligned} y_1 &= 1 \quad \int\int|\varphi(\mathrm{x})|\mathrm{dx}^2+\int\int|\varphi(\mathrm{x})|dx^2\int\int|\varphi(\mathrm{x})|\mathrm{dx}^2 - \\ &\int\int|\varphi(\mathrm{x})|\mathrm{dx}^2\int\int|\varphi(\mathrm{x})|\mathrm{dx}^2\int\int|\varphi(\mathrm{x})|\mathrm{dx}^2+...... \\ y_2 &= x \quad \int\int x|\varphi(\mathrm{x})|\mathrm{dx}^2+\int\int|\varphi(\mathrm{x})|dx^2\int\int x|\varphi(\mathrm{x})|\mathrm{dx}^2 \\ &\int\int|\varphi(\mathrm{x})|\mathrm{dx}^2\int\int|\varphi(\mathrm{x})|\mathrm{dx}^2\int\int x|\varphi(\mathrm{x})|\mathrm{dx}^2+...... \end{aligned} \qquad (7)$$

and these sequences define the oscillating solutions (5) we have found earlier

$$y_1 = \cos_{|\phi(x)|} x \approx \cos(x\sqrt{|\phi(x)|}) \quad (8)$$
$$y_2 = \sin_{|\phi(x)|} x \approx \frac{1}{\sqrt{|\phi(x)|}} \sin(x\sqrt{|\phi(x)|})$$

Based on solutions we have obtained, it can be concluded the following:
There is a unique, real, so called "function of frequency"

$$g(x) = x\sqrt{|\phi(x)|} \quad (9)$$

so that the solutions (6) and (8) can be united under common symbol

$$y_{1/2} = e^{\pm ix\sqrt{\phi(x)}} = e^{\pm ig(x)} \quad (10)$$

which includes both cases ϕ<0, ϕ>0.

The amplitudes of particular integrals are not the same, but they depend on ϕ(x), in the most different ways. This means that if we use only real numbers, the oscillating solutions of the equation (1) should be searched, in the circumstances which are general enough in the form of functions:

$$y_1 = F(x)\cos g(x)$$
$$y_2 = G(x)\sin g(x) \quad (11)$$

where the functions g(x), F(x) and G(x) must beforehand satisfy some minimum conditions of continuous differentially in order to get the solution type (11). As for periodicity of one or both solutions, some additional conditions should be discovered and proved.

Regarding the general nature of random F, G and g, we shall examine possibility of periodicity of, at least, some part of the solution (11), i.e. the possibility of g(x) and sin g(x) to be periodic functions.

Let us consider cos g(x) to be periodic function. In this case following the definition of periodicity, we have:

$$\cos g(x+\omega) = \cos g(x)$$

and therefore, according the very definition of the elementary function cos α, it follows:

$g(x+\omega) = g(x) + 2k\pi$; k=0,1,2,3…

which further gives

$$g(x+\omega) - g(x) = 2k\pi$$

If here we apply the mean value theorem, we will get

$$g'(\xi)\cdot[(x+\omega) - x] = 2k\pi$$

or

$$g'(\xi)\cdot\omega = 2k\pi$$

Number ω which has been obtained for k≈1 is called a minimal period of the function cos x and in that case we have:

$$g'(\xi) = \frac{2\pi}{\omega} = Const = \alpha$$

However, if x changes and increases, then the interval [0, x] is also changed, including the position of the mean point ξ, which also makes g′(ξ) a variable value together with x. Based on this, when compared to the above formula, it leads to the contradiction that ω=Const. This also gives the following differential equation:

$$g'(\xi) = \alpha, \qquad dg = \alpha d\xi, \qquad g = \alpha\xi + \beta$$

where g(ξ) must be a linear function, i.e. if we, at this moment, neglect the unimportant constant β, there must be

$$g(x) = \alpha x \tag{12}$$

This implies

Theorem 2.

The function cos g(x) (and same goes for sin g(x)) is a periodic function, if g(x) is basically only a linear function of an argument x given with (12).

(Note: If x is not independent variable, then g′(x) . x′(t) appears in the derivative and in this case different form of the theorem becomes valid.

Therefore, we specialize the general solutions (11) with (12):

$$\begin{aligned} y_1 &= F(x)\cos\alpha x \\ y_2 &= G(x)\sin\alpha x \end{aligned} \tag{13}$$

Based on the first derivatives of the presumed solutions (13) we get:

$$y_1' = F'\cos\alpha x - \alpha F\sin\alpha x$$
$$y_2' = G'\sin\alpha x + \alpha G\cos\alpha x$$

Hence we get the Wronscian of the adequate second order linear homogenous differential equation with solutions (13)

$$\begin{aligned} W(y_1, y_2) &= y_1' \cdot y_2 - y_1 \cdot y_2' \\ &= -\alpha FG + (F'G - G'F)\sin\alpha x\cos\alpha x \end{aligned} \tag{14}$$

Having obtained (14) and in order to make equation (1) possible, where ϕ(x) would be continuous coefficient, it should be W(x)≠0. Therefore, we get the following discussion:

1° If we want F and G to be periodic and to have sinαx or cosαx as a multiplier, then there should be W=0 in zeros of these functions sinαx and cosαx, which means that the equation composed of (13) cannot have continuous coefficients.

2° In all other points which are isolated in which W=0, differential equation with solutions y_1, y_2 of type (13) can be composed.

$$Wy'' - A(x)y' + B(x)y = 0 \tag{15}$$

But it will have neither continuous coefficients, nor some sort of canonical form (1) which would be continuous.

This implies:

Theorem 3.

Functions F(x) and G(x) in (3) can be periodic, but in order to make equation (1) with continuous coefficient ϕ(x) from (3), they cannot have the same or commensurable period with sinαx, cosαx.

In this case (13) implies that y_1, y_2 have double roots due to factors $(\sin\alpha x)^2$ or $(\cos\alpha x)^2$, which is not possible owing to the well known theorems on the solutions existence of the differential equations with continuous coefficients.

First, based on other derivatives of the presumed solutions:

$$y_1'' = F''\cos\alpha x - 2\alpha F'\sin\alpha x - \alpha^2 F\cos\alpha x$$
$$y_2'' = G''\sin\alpha x + 2\alpha G'\cos\alpha x - \alpha^2 G\sin\alpha x$$

we set the condition that (1) is satisfied with the pair y_1, y_2. Thus, we indirectly have:

$$F''\cos\alpha x - 2F'\alpha\sin\alpha x - F\cos\alpha x\cdot[\alpha^2 + \phi(\text{x})] = 0$$

$$G''\sin\alpha x - 2G'\alpha\cos\alpha x - G\sin\alpha x\cdot[\alpha^2 + \phi(\mathrm{x})] = 0$$

and we find f through F, and then G.

$$\phi(\mathrm{x}) = \frac{\mathrm{F}''}{\mathrm{F}} - 2\alpha\frac{F'}{F}tg\alpha x - \alpha^2 \qquad (16)$$

$$\phi(\mathrm{x}) = \frac{\mathrm{G}''}{\mathrm{G}} - 2\alpha\frac{G'}{G}ctg\alpha x - \alpha^2 \qquad (17)$$

Based on this a condition that F and G are part of the solution is applied

$$F''G - FG'' = 2\alpha[G'Fctg\alpha x + F'Gtg\alpha x] \qquad (18)$$

For F(x) ≅G(x) the Wronskian is W=-α·F^2 and condition (18) gives

$$0 = 2\alpha FF'[ctg\alpha x + tg\alpha x] = 0$$

which is fulfilled only for F′=G′=0, i.e. F=G=Const, while (16) and (17) also give f(x)=-α^2=Const, otherwise these would be opposite conditions. It follows that there does not exist <u>the canonical equation</u> which would be satisfied by the functions (13), where F(x) and G(x) are effective functions of x, but these are only constants, i.e. (13) are harmonic oscillations y′′=-α^2.y. However, it does not mean that there is not a <u>full</u> second order linear homogenous differential equation which would have solutions (13). Therefore, we shall observe this case separately.

III. Let us see if there exist the second order equation with the solutions

$$\begin{aligned} y_1 &= F(x)\cos\alpha x \\ y_2 &= F(x)\sin\alpha x \end{aligned} \qquad \text{(F=G)} \qquad (19)$$

i.e. <u>the equally amplitudinal periodic solutions</u>. By forming equation following the Liouville's method

$$\begin{vmatrix} y'' & y' & y \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix} = 0$$

we easily get (15) where

$$W = -\alpha F^2; \quad a(x) = -\frac{2F'}{F}, \; b(x) = \frac{2F'^2 - FF'' + \alpha^2F^2}{F^2}$$

Therefore it follows

***Theorem 4**.*

The equation

$$y'' - \frac{2F'}{F}y' + (\alpha^2 + \frac{2F'^2 - FF''}{F^2})y = 0 \qquad (20)$$

has solution (19) with the same amplitude F(x).

However, if we try to find the canonical form of (20), by known replacement

$$y = e^{-\frac{1}{2}\int(-\frac{2F'}{F})dx}. \quad z = e^{\ln|F|}. \quad z = F\cdot z, \; F>0$$

we can easily get the canonical form

$$z'' + B(x)z = 0$$

where

$$B = b \quad \frac{a''}{2} \quad \frac{a''}{4} = \alpha^2 + \frac{2F'^2 \quad FF''}{F^2} \quad \frac{1}{2}(\; 2\frac{F'}{F})^1 \quad \frac{1}{4}(\quad \frac{2F'}{F})^2 = \alpha^2$$

i.e.

$z'' + \alpha^2 . z = 0$ (21)

These are harmonic oscillations, whose solutions are z_1=cos αx, z_2=sin αx, and therefore $y_{1,2}$=F.z, which gives

$y = F(x)[C_1 \cos \alpha x + C_2 \sin \alpha x]$ (22)

Thus the equation (21) can be considered not only canonical, but also characteristic for the entire linear homogenous differential equation (20), where F(x) is twice continuous differentiable function. This directly implies

Theorem 5.

a) If F(x) is monotonic function, it defines the solutions in the form of oscillating functions with increasing amplitude F(x) and equally distant zeros.
b) If F(x) is non-monotonic, but positive function, then (20) has solutions in the form of oscillating functions with variable amplitude F(x)
c) If F(x) is also an oscillating function, then the solutions (20) are double oscillating functions, also with variable amplitude,
d) If F(X) is periodic function, then the solution (22) is also periodic, if the period F is commeasurable with period $2\pi/\alpha$.
e) If F(x) is periodic function with second period, then (22) are oscillating solutions of the equation (20), but with two periods.

IMPORTANT COROLLARY: The case d) of the Theorem 5. is considered to be special case of an important and basic Flouguet's theorem, meaning that the solutions of the second order linear homogenous differential equation are oscillating functions of the following type (23)

$$y_1 = \cos \alpha x \cdot p(x)$$
$$y_2 = \sin \alpha x \cdot p(x) \qquad (23)$$

or as it is written in the classic form

$y_{1/2} = e^{\pm i\alpha x} \cdot p(x)$, *p* – periodic

EXAMPLES

1. The equation $y'' - 2ctgx \cdot y' + (\alpha^2 + \frac{1+\cos^2 x}{\sin^2 x})y = 0,$
where F(x)=sin x, has the following solution

$$y_1 = \sin x \cdot \cos \alpha x$$
$$y_2 = \sin x \cdot \sin \alpha x$$

2. The equation $y'' - 2y' + (1+\alpha^2)y = 0,$
where F(x)=e^x, has the following solution.

$$y_1 = e^x \cdot \cos \alpha x$$
$$y_2 = e^x \cdot \sin \alpha x$$

3. The equation $y'' + \frac{2}{\sin x \cos x} y' + [\alpha^2 + \frac{2(\sin x - 2\cos^2 x)}{\sin^3 x \cos^2 x}]y = 0,$
where F(x)=ctg x, has the following solution.

$$y_1 = ctgx \cdot \cos \alpha x$$
$$y_2 = ctgx \cdot \sin \alpha x$$

4. The equation $y''+\frac{2\sin 2x}{1+\cos^2 x}y'+[\alpha^2+\frac{2\sin^2 2x+4\cos^2 x(1+\cos^2 x)}{(1+\cos^2 x)^2}]y=0,$

where F(x)=1+cos^{2}x, has the following solution

$y_1=(1+\cos^2\mathrm{x})\cdot\cos\alpha x$

$y_2=(1+\cos^2\mathrm{x})\cdot\sin\alpha x$

5. The equation $y''+2xy'+(\alpha^2+\frac{2+\ln x}{x^2\ln x})y=0,$

where F(x)=lnx, has the following solution

$y_1=(\ln x)\cdot\sin\alpha x$

$y_2=(\ln x)\cdot\cos\alpha x$

6. The equation $y''-y'(\alpha ctg\alpha x)y^1+(\alpha^2 ctg^2\alpha x)=0,$

where F(x)=sinαx has the following solution

$y_1=\sin\alpha x\cdot\cos\alpha x$

$y_2=\sin^2\alpha x$

The solutions can be designed in different ways, even with multiple zeros, which will give the poles of the second order in the coefficients (as we have already shown it). Nevertheless, it is a job of tabular equations as given in the manual by E. Kamke.

Zeros of the second order are possible in the solutions, because differential equation does not have continuous coefficients.

The characteristics of the solution (8) of the Hill's equation (1) that it has solution in different amplitudes in general case, makes it more important than III the case of double amplitude solutions (11), where F(x) ≠ G(x), although both functions are continuously differentiable.

The natural case of solutions is natural to search through (13), where g(x)=αx. Also, all formulas such as (14), (15), (16), (17) and (18) are valid.

Without special difficulties, by applying the usual technique, we form he equation (15) with solutions (13), where the Wronskian W(y_1,y_2)≠0. It goes:

$$y''+\frac{1}{W}[-(F''G-G''F)\sin\alpha x\cos\alpha x+2\alpha(F'G\sin^2\alpha x+FG'\cos^2\alpha x]y+$$
$$+\frac{1}{W}\{\sin^2\alpha x[(G''-\alpha^2G)\alpha F-2\alpha F'G']+ \qquad (24)$$
$$+\cos^2\alpha x[(F''-\alpha^2F)\alpha G-2\alpha F'G']+$$
$$+\sin\alpha x\cos\alpha x[(F''-\alpha^2F)G'-(G''-\alpha G)F'+2\alpha^2(-F'G+G'F)]\}=0$$

The equation (24) is a full second order linear homogenous differential equation and it is not the Hill's equation. However in order to be connected to the equation (1), we have seen that it is necessary to fulfill the condition for the connection between F, G and ϕ, i.e. $\phi=\phi$(F,G), and that is the condition (18). Nevertheless, we also see that it is exactly the coefficient in front of y in the last equation (24), or the coefficient a(x) in (15) is

$$a(x)=-\frac{A(x)}{W}=-\frac{dW/dx}{W}=0$$

according to the Liouville's formula, and it means that W=Const=C, i.e.

$$C=-(F''G-G''F)\sin\alpha x\cos\alpha x+2\alpha(F'G\sin^2\alpha x+FG'\cos^2\alpha x) \qquad (25)$$

And this means that now (24) does not have a member with y′, i.e. it is a canonical form.
From (25) we have further elimination

$$\sin\alpha x\cos\alpha x=\frac{-C+\alpha FG}{F'G-G'F}=P(F,G) \qquad (26)$$

where the constant C depends on initial conditions $\{y_1^{\,0},y_1^{\,\prime o},y_2^{\,0},y_2^{\,\prime 0}\}$. Based on (26), we find by use of trigonometric formulae

$$\cos\alpha x=\sqrt{\frac{1}{2}[1+\sqrt{1-4(\frac{C+\alpha FG}{F'G-FG'})^2}]}=N(F,G) \qquad (27)$$

$$\sin\alpha x=\sqrt{\frac{1}{2}[1-\sqrt{1-4(\frac{C+\alpha FG}{F'G-FG'})^2}]}=M(F,G) \qquad (28)$$

And from (24) we get the canonical form, but this time without functions sin αx and cos αx

$$\begin{aligned}&y''+\frac{\alpha}{C}\{M^2[G''-\alpha^2G)F-2F'G']+\\&+N^2[(F''-\alpha^2F)G-2F'G']+\\&\frac{1}{2}P[(F''-2\alpha^2F)G'-(G''-\alpha^2G)F'+\\&+2\alpha^2(-F'G+G'F)\}y=0\end{aligned} \qquad (29)$$

This is an equation of the type $y''+\psi(F,G,\alpha)y=0$ where α is an independent constant, and C depends on initial conditions. It has solutions (13) which are, as a rule, oscillating.
Let us have now F and G periodic functions with the periods 2π. Then the coefficient in (29) is obviously the periodic function, because M, N and P are such, including all derivatives, while α is an independent constant. Then this is a Hill's equation (1).
This implies:

Theorem 6.
The equation (29) is a Hill's equation (1) if F and G, which form $\psi(x)$ in (29), are periodic functions with some other periods $\omega=2\pi/\alpha$. The solutions of (29) are the functions

$$\left.\begin{aligned}y_1&=F(x)\cos\alpha x\\ y_2&=G(x)\sin\alpha x\end{aligned}\right\}\quad \text{F, G – periodic}$$

or as it Flouquet writes in its classical form

$$\left.\begin{aligned}y_1&=e^{i\alpha x}\cdot p_1(x)\\ y_2&=e^{-i\alpha x}\cdot p_2(x)\end{aligned}\right\}\quad p_1, p_2 \text{ – periodic}$$

Note: Flouquet in its work [1], gives the method for determination of α, but we provide the way in which ϕ(x) in (1) depends on amplitudes, i.e. on periodic functions F and G.

CONCLUSION

Neither did we completely and squarely solve the Hill's equation (1), nor Flouquet did that, because, in general case, it is impossible (although Berkovic considers it still unproved [6]). We provide the way in which ϕ(x) in (1) depends on periodic parts of the solution. If F is given, then ϕ is a periodic function of G, or the opposite, if G is given, then ϕ is a periodic function of F. This means that if ϕ is a given complex periodic function in the Hill's

equation, by method of "decomposing" ϕ on F and G, we get F and G that are soluble or known equations, which makes (1) possible to be solved from (29).
Example: In this sense, obviously, the simplest choices are $F'' - \alpha^2 F = 0$, or $G'' - \alpha^2 G = 0$, or similar. It is possible to make table with numerous results of concrete Hill's equation, even in closed form, as in the mentioned examples.

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