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A BISECTORIAL MODEL FOR ECONOMIC GROWTH

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Abstract: The paper emphasizes two analytical solutions for the bisectorial model introduced by Uzawa for economic growth. In the classical case, it assures a sufficient condition for the uniqueness of equilibria on the production factors market. The solution in the case of convex savings emphasizes a new point of view that considers a special relation of convex type between the propensity to save out of wages and the propensity to save out of profits. A locally stable solution for the evolution equation that describes the dynamics of the capital is also provided.

1. PRELIMINARIES AND NOTATIONS

Let us consider an economy that produces, by means of capital and labor, two goods, a commodity for consumption and another commodity for investment. The bisectorial model described by Uzawa in [3] and [4] is an extension of the Solow-Swan neoclassical growth model, which had as principal idea to give a less restrictive relation between capital and income. The Uzawa model is characterized by two inputs and two outputs, one of the latest being also an input.

In what follows we will describe the economic relations of the model. The capital stock $K(t)$ and the labor $L(t)$ are used as inputs. We consider the production function given, at any moment t, by the relation

$$Y(t) = F(K(t), L(t)),\ \forall t \geq 0,$$

both for the consumption sector, denoted Y_c, as well as for the sector of accumulation or investments, denoted Y_i. In what follows, the index c or i will indicate if the consumption or the investments sector is concerned. We suppose that there is no depreciation of the capital, all technological progress is endogenous and the level of population is maintained constant. If $p(t)$ denotes the price at moment t of the investment goods relative to the consumption goods, the income will be given by the relation

$$Y(t) = p(t)\, Y_i(t) + Y_c(t),\ \forall t \geq 0 .$$

The properties of the production function are:

(i) function F is homogenous and linear, which indicates constant productivity

(ii) function F is positively defined and the lack of one of the factor implies a null value

$$F(K(t), 0) = 0 \text{ and } F(0, L(t)) = 0,\ \forall t \geq 0.$$

(iii) function f has positive partial derivatives of first order, representing the marginal productivity of capital, respectively of labor

$$F'_K(t) = \frac{\partial F(K,L)}{\partial K} > 0 \text{ and } F'_L(t) = \frac{\partial F(K,L)}{\partial L} > 0,\ \forall\ K, L > 0,\ \forall t \geq 0$$

(iv) function f has negative partial derivatives of second order

$$F''_{KK}(t) = \frac{\partial^2 F(K,L)}{\partial K^2} < 0,\ F''_{LL}(t) = \frac{\partial^2 F(K,L)}{\partial L^2} < 0,\ F''_{KL}(t) = \frac{\partial^2 F(K,L)}{\partial K \partial L} < 0,\ \forall\ K, L > 0, \forall t \geq 0$$

(v) $\lim\limits_{K \to 0} F(K(t), L(t)) = \infty$ and $\lim\limits_{K \to \infty} F(K(t), L(t)) = 0,\ \forall L > 0,\ \forall t \geq 0.$

Remark 1.1. From relations (i) and (iv) it is obvious that following statements hold, similar to the Inada conditions [2]:

(a) $\lim_{K\to 0} F'_L(K(t), L(t)) = 0$ and $\lim_{K\to\infty} F'_L(K(t), L(t)) = \infty$, $\forall L > 0$, $\forall t \geq 0$

(b) $\lim_{L\to 0} F'_K(K(t), L(t)) = 0$ and $\lim_{L\to\infty} F'_K(K(t), L(t)) = \infty$, $\forall K > 0$, $\forall t \geq 0$.

In what follows, the lowercases will indicate the characteristics defined per capita, where every element is divided by the level of labour. Let us consider that the production function per capita will depend only on the capital $f = f(k)$. Some variables and relations involved in the bisectorial model are:

- the price on the labor market: $p^{lm}(t) = \dfrac{dY_c(t)}{dL_c(t)} = f_c(t) - k_c(t)f_c'(t)$
- the price on the capital market: $p^{cm}(t) = \dfrac{dY_c(t)}{dK_c(t)} = f_c'(t)$
- the change in the stock of capital is the sector's output: $Y_i(t) = \dfrac{dK(t)}{dt}$
- the labor growth rate: $r_L(t) = \dfrac{1}{L(t)}\dfrac{dL(t)}{dt} = \lambda$
- the capital growth rate: $r_C(t) = \dfrac{1}{K(t)}\dfrac{dK(t)}{dt} = \dfrac{Y_i(t)}{K(t)}$
- the labor level per capita in the two sectors: $m_c(t) = \dfrac{L_c(t)}{L(t)}$, $m_i(t) = \dfrac{L_i(t)}{L(t)}$
- the equilibrium in the labor market: $m_c(t) + m_i(t) = 1$
- the equilibrium in the capital market: $k_c(t)m_c(t) + k_i(t)m_i(t) = k(t)$.

Further, we will not always mention the time variable t, even if every element is time-dependent. We will denote by h the ratio between wage and profit. Let us define the function

$$h: [0, \infty) \to [0, \infty),\ h(k) = \frac{f(k)}{f'(k)} - k > 0,$$

the definition of this function being deduced from the relations that give the prices on the labor, respectively capital market. Function h has following properties:

$$h'(k) = -\frac{ff''}{(f')^2}(k),\ \lim_{k\to 0} h(k) = 0,\ \lim_{k\to\infty} h(k) = \infty.$$

Remark 1.2. Function h is invertible, its inverse being denoted $k: [0, \infty) \to [0, \infty)$, with the properties that it is nondecreasing as

$$k'(h) = \frac{(f')^2}{ff''}(h) > 0 \text{ and } \lim_{h\to 0} k(h) = 0,\ \lim_{h\to\infty} k(h) = \infty.$$

To express the growth equation, we assume that the per capita value of the capital must be constant, which characterize the steady-state and which means

$$k(t) = \frac{K(t)}{L(t)} = \text{const.}, \forall t \geq 0,$$

and which can be further written, according to the labor growth rate and the capital growth rate

$$r_K(t) = r_k(t) + r_L(t),\ \forall t \geq 0.$$

Thus, the differential equation that describes the evolution of the bisectorial model and the dynamic of the capital is

$$k'(t) = y(t) - \lambda k(t),\ \forall t \geq 0,$$

where $y(t) = \frac{Y(t)}{L(t)}$ is the income per capita of the investments sector.

2. THE MAIN RESULTS

In this section we will present two solutions for the evolution equation that describes the bisectorial model, depending on the hypothesis concerning the savings type.

2.1. The solution in the classical case

We will consider, as in [3], that there are no savings for the workers, which consume all their incomes and that the business owners save all their gains. The demand for consumption good at moment t is defined by $p^{lm}(t)L(t)$ and the demand for consumption good at moment t by $p^{cm}(t)K(t)$. The equilibrium relations at moment t are given by

$$Y_c(t) = p^{lm}(t)L(t) \text{ and } p(t)Y_i(t) = p^{cm}(t)K(t).$$

According to the per capita relations to the equilibrium relations on the capital, respectively labor market and to the definition of functions $k \to h(k)$ and $h \to k(h)$ given on $[0, \infty)$, we obtain

$$\frac{1}{k}\, k'(h) = \frac{h}{k_c(k_c+h)}\, k_c^{'}(h) + \frac{1}{k_i+h}\, k_i^{'}(h) + \frac{k_c-k_i}{(k_c+h)(k_i+h)}.$$

The sign of $k'(h)$ depends on the sign of the last term of the previous relation, so, as in [3], if $k_c(h) > k_i(h)$, for all $h \in [\min\{h(k_c), h(k_i)\}, \max\{h(k_c), h(k_i)\}]$, which assures a sufficient condition for the uniqueness of equilibria on the production factors market. It is then obtained that, considering Uzawa's capital-intensity asumptions,

$$k'(h) > 0, \text{ for all } h \in [\min\{h(k_c), h(k_i)\}, \max\{h(k_c), h(k_i)\}].$$

2.2. The solution in the case of convex savings

There exist, for the workers, as well as for the business owners a given coefficient of the weights, respectively gains for savings.

As a new point of view, different from the classical one presented in the papers that studied Uzawa's model, we will consider a special relation between the propensity to save out of wages, denoted α and the propensity to save out of profits, denoted $1-\alpha$. The demand for investments goods at any moment t is given by

$$\alpha p^{lm}(t)L(t) + (1-\alpha)p^{cm}(t)K(t)$$

and the demand for consumption goods at any moment t is given by

$$(1-\alpha)p^{lm}(t)L(t) + \alpha p^{cm}(t)K(t).$$

We obtain the relations that describes the equilibria, in per capita form, at any moment t

$$p(t)y_i(t) = \alpha p^{lm}(t) + (1-\alpha)p^{cm}(t)k(t)$$

$$y_c(t) = (1-\alpha)p^{lm}(t) + \alpha p^{cm}(t)k(t).$$

The evolution equation that describes the dynamic of the capital can be written as

$$k'(t) = [\alpha h(t) + (1-\alpha)k(t)]\, f_i^{'}(k_i(t)) - \lambda k(t),\ \forall t \geq 0.$$

The equilibrium condition $k'(t) = 0$, $\forall t \geq 0$, implies

$$\lambda = \frac{[\alpha h + (1-\alpha)k(t)] f_i^{'}(k_i(t))}{k(t)},\ \forall t \geq 0.$$

According to the relations that define the labor level per capita in the consumption and in the investments sectors and according to the equilibrium in the labor and in the capital market, following system with variables $k(t)$ and $m_i(t)$:

$$\begin{cases} k(t) + m_i(t)[k_c(t) - k_i(t)] - k_c(t) = 0 \\ (1-\alpha)k(t)f_i^{'}(k_i(t)) - m_i(t)f_i(t) + \alpha h f_i^{'}(k_i(t)) = 0 \end{cases}$$

We obtain

$$k\ (t) = \frac{\alpha h[k_i(t) - k_c(t)] + k_c(t)[k_i(t) + h]}{\alpha k_i(t) + (1-\alpha)k_c(t) + h},\ \forall\, t \geq 0.$$

We remark that, according to our hypothesis, one of the Drandakis conditions (see [1]) concerning the inequality between the propensities for savings holds. When the second condition for the per capita level of the capital in the consumption and in the investment sectors is also true, we have the next result.

Proposition 2.1. *If $k_c\ (t) \geq k_i\ (t)$, $\forall\, t \geq 0$, then the capital-labor ratio k is a locally stable solution of the evolution equation.*

Proof. According to the formula obtained for k, following equivalent relations hold for all t

$$\frac{d(k'(t))}{dk} = \left[\alpha\left(\frac{dh}{dk} - 1\right) + 1\right] f_i^{'}(t) + \left[\alpha(h - k(t)) + k(t)\right] f_i^{''}(t) \frac{dk_i}{dh}\frac{dh}{dk} - \lambda$$

$$\frac{d(k'(t))}{dk} = \alpha \frac{dh}{dk} f_i^{'}(t) + (1-\alpha) f_i^{'}(t) - \frac{f_i^{'}(t)}{f_i(t)}\left[\alpha(h - k(t)) f_i^{'}(t) + k(t) f_i^{'}(t)\right]\frac{dh}{dk} - \lambda k(t).$$

Further, we have

$$\frac{d(k'(t))}{dk} = \left[(\alpha - m_i(t))\frac{dh}{dk} - \alpha\frac{h}{k}\right] f_i^{'}(t) + \frac{m_i(t)}{k(t)} f_i(t) - \lambda.$$

As the sign of the second order derivative depends on the sign of the first term of the previous relation, we can conclude that for

$$\alpha\left(\frac{dh}{dk} - \frac{h}{k}\right) < m_i(t)\frac{dh}{dk},\ \forall\, t \geq 0,$$

The solution k is then locally stable, which ends the proof. □

Corollary 2.2. *Let k* be the steady state of k. If*

$$\alpha \leq \lambda \frac{k^*}{f_i(k^*)}$$

then the steady state capital-labor ratio is locally stable.

Remark 2.3. Without any loss of generality we can also assume, similarly as in Uzama's savings assumption, that α = 0.5, a condition that is sufficient for the uniqueness of the market equilibrium.

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