CREATING 2D IRREGULAR SHAPE USING FUNCTION BÉZIER

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Abstract Geometrical shapes commonly used in Computer Aided Design (CAD) systems can be defined by several points obtained through the digitizing process. The coordinates of the points situated between two nodes can be approximated through both analytic and graphic methods, with Bézier curves. Thus, the analytic expression of the curve that approximates the points will be a Bezier function. The graphical form will be represented by a curve that crosses all the co-ordinates of the digitized points, without bringing any mutations of the initial curve. For case study on analyzing creating of the footwear parts

1. IRREGULAR SHAPES: COMPUTER MODELING

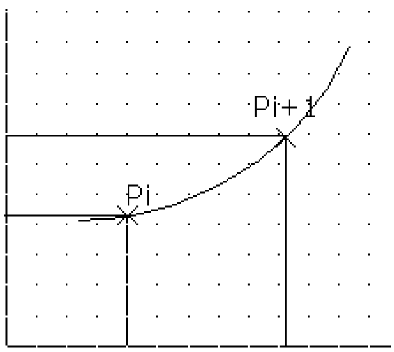


Fig.1, Generating the smoothest interpolating curve that crosses most of the initial points.

The modeling geometric forms on a computer can be reduced to a problem of numeric approximation: one curve, drawn by a specific equation, must fit a certain number of points (*see fig. 1*). However, generally speaking, irregular forms used in design problems cannot be described by a simple function like *y=f(x)* [1], [2].

In order to acquire the analytic expression of the interpolating function and its graphic form, the problem will be approached in the following manner:

- ❑ We will take a finite set of points in the same geometrical plan:

$$(x_0, y_0), (x_1, y_1)\ldots (x_{n-1}, y_{n-1})$$

- ❑ We have to generate the smoothest interpolating curve that crosses most of these points.

We call *interpolation* the analytic expression of a geometrical shape that is numerically encoded.

Interpolation is a particular case of a more general problem of mathematical approximation [2]. The function *f(x)* will be approximated, by interpolation, with the function g(τ), which is defined by a finite sum of simple functions $\Psi_i(x_i)$, as in the following expression:

$$\mathbf{g(\tau)=\sum c_i\psi_i(\tau)},\ i\in[1,n] \qquad (1)$$

In order to be able to determine the n constants $c_1, c_2, c_3, \ldots c_n$, we have to identify several restrictive conditions for the $g(\tau)$ function. The following restrictive conditions are commonly used in the theory of approximation:

1. Interpolation constraints:

$$\mathbf{g(\tau_i)=f(x_i)},\ \forall \in i\ [1,n] \quad (2)$$

2. Mixture of interpolation and constraints:

$$\begin{aligned} &\text{a) } \mathbf{g(\tau_i)=f(x_i)} \quad \forall i \in [i:k<n] \\ &\text{b) } \mathbf{g'(\tau_1)=f'(x_1)} \text{ and } \mathbf{(x_k)=f'(x_k)} \\ &\text{c) } \mathbf{g(\tau)} \text{ is a twice differential function} \end{aligned} \quad (3)$$

3. Various constraints:

$$\|f-g\|=\min\{\|f-h\| \,/ h \in \text{distance } (\psi_1,\psi_2,\ldots.\psi_n)\} \quad (4)$$

i.e. constants $c_1, c_2, c_3, \ldots c_n$ should be chosen in such a way as to allow that the minimum of all possible functions $\|f\text{-}g\|$ should be obtained of the set of all possible linear combinations:

2. DEFINING INTERPOLATING POLYNOMS

However, theory cannot be directly applied in the computer-aided graphics, and this is why we have to approach the interpolation problem from a parametric point of view.

The parametric equations used in interpolation are actually polynomial equations, usually bicubic, described as:

$$g(t)=at^3+bt^2+ct+d$$

Thus, we will approximate the set of points $(x_0, y_0), (x_1, y_1) \ldots (x_{n-1}, y_{n-1})$ by the aid of two variables $x(t)$ and $y(t)$, defined by two parametric interpolation equations:

$$\begin{aligned} x(t)&=a_x t^3+b_x t^2+c_x t+d_x \\ y(t)&=a_y t^3+b_y t^2+c_y t+d_y \end{aligned} \quad (5)$$

In order to do that, we will attach two supplementary systems of co-ordinates – *tOx* and *tOy* – to the present one – *xOy*. We select the variation domain of parameter t. Then, we solve the two independent problems of theoretical interpolation for the two variables, x and y. This will allow the determination of the four coefficients from the following restrictive conditions:

1. The value of the polynom in the nodes must be the same with its numeric value:

$$x(t_i)=x_i,\ y(t_i)=y_i \quad (6)$$

where i=0… n and x_i and y_i are nodes: (x_0, y_0), (x_1, y_1)… (x_n, y_n)

1. The vector that is tangent to the curve in a node must be the same as the vector of the initial curve. If the components of the vector tangent to the interpolating curve are:

$$x'(t)=3a_xt^2+2b_xt+c_x$$
$$y'(t)=3a_yt^2+2b_yt+c_y \qquad (7)$$

then the restrictive conditions for the limit will be:

$$x'(t_i)=l_i \text{ and } y'(t_i)=m_i \qquad (8)$$

where l_i and m_i are the values of the incline of the tangents in the nodes, as determined in the tOx and tOy co-ordinates systems.
The mathematical representation of the equation (5), (6), (7) and (8) leads to the following conclusions:

1. In order to obtain the interpolating polynoms between two nodes (x_i, y_i) and (x_{i+1}, y_{i+1}), we have to solve a system of 8 equations with 8 unknown values. The equations will be determined this way:
- Relation (5) combined with restriction (6) will allow us to obtain 4 of the equations in the system;
- Relation (7) combined with restriction (8) will allow us to obtain the remaining 4 equations.

2. The way we choose the variation domain of the parameter t and the way we choose the direction of the tangents in the nodes will determine the manner in which we theoretically solve the interpolation proces.

3. THEORETICALLY SOLVING THE INTERPOLATION BÉZIER CURVES

Without affecting the general area of an interpolation problem, the interpolating Bézier polynoms derive from the following conditions[1], [2]:

C1. We take the interval [0,1] as a variation domain for parameter t. In this case, the conditions for the limit (relations (5) and (6)) will be:

$$x(0)=d_x \qquad y(0)=d_y$$
$$x(1)=a_x+b_x+c_x+d_x \qquad y(1)=a_y+b_y+c_y+d_y \qquad (9)$$

C2. The directions of the tangents in the nodes will be defined as the inclination of the tangent led in every node. For example, in the (x_i,y_i) node, they will be calculated with the following relations:

$$l_i=m(x_i-x_{c1}) \qquad m_i=m(y_i-y_{c1})$$
$$l_{i+1}=m(x_{i+1}-x_{c2}) \qquad m_{i+1}=m(y_{i+1}-y_{c2}) \qquad (10)$$

or:

for t=0 $c_x = m(x_i - x_{c1}) \quad c_y = m(y_i - y_{c1})$

for t=1 $3a_x + 2b_x + c_x = m(x_{i+1} - x_{c2}) \quad 3a_y + 2b_y + c_y = m(y_{i+1} - y_{c2})$ (11)

where m, known as a **shape factor** in the literature, usually takes the value 3, x_i, y_i and x_{i+1}, y_{i+1} are the co-ordinates of the nodes (the extreme point of the curves), x_{c1}, y_{c1} and x_{c2}, y_{c2} are the co-ordinates of the two points that belong to the tangents to the Bézier curve and they are called **points of control**.

Relations (9) and (11) of the two conditions lead to a system of 8 equations with 8 unknown values, with the following solutions:

$$\begin{aligned} d_x &= x_i & d_y &= y_i \\ c_x &= 3(x_{c1} - x_i) & c_y &= 3(y_{c1} - x_i) \\ b_x &= 3(x_{i+1} - x_i) - c_x & b_y &= 3(y_{i+1} - y_i) - c_y \\ a_x &= x_{c2} - x_i - c_x - b_x & a_y &= y_{c2} - y_i - c_y - b_y \end{aligned} \quad (12)$$

Relation (12) represents the mathematical expression of the coefficients of the bicubic polynomial Bézier functions.

If we analyse the two conditions, the <u>conclusions</u> will be as follows:

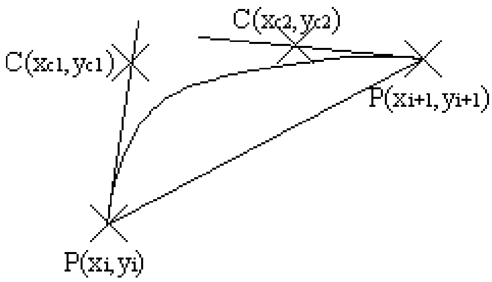


Fig. 2, Defining the Bézier curves

1. A Bézier curve is defined by four points:

- two fixed points on the Bézier curve (nodes), that are fixed;
- two other intermediate points, that belong not to the curve, but to its tangents. The two points are called **control points** and are positioned on the tangents of the curve led in the nodes($P(x_i,y_i)$, $P(x_{i+1},y_{i+1})$, *see fig. 2*).

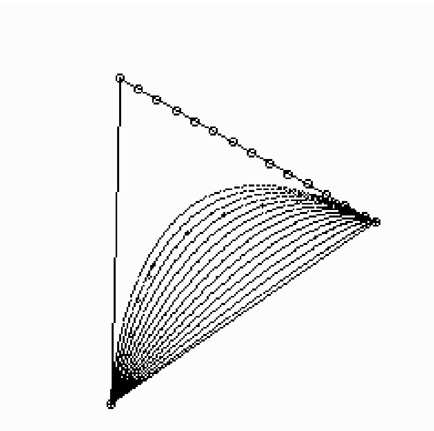


Fig. 3, Between two nodes we can define many Bézier curves

2. Between two nodes we can define many Bézier curves, related to the position of the control point on the tangent *(see fig. 3)*. This makes it possible, in a CAD work session, to draw several Bézier curves and choose the convenient one – the one that approximates a set of points between the two nodes with the highest precision.

3. Most of the points that define a Bézier curve and their graphic display are obtained by giving various values (from 0 to 1) to the **t** parameter for the curve described with equations (10) whose parameters had been determined by relations (12).

4. The graphical form of the Bézier curve will be determined by the position of the nodes (fixed points) and the co-ordinates of the control points on the two tangents led to the curve. Depending on their position, we will obtain either concave or a convex curve, in any case a curve with a singe turning point (*see picture 4*).

4. ORIGINAL SOFTWARE PRODUCT USING BÉZIER CURVES FOR INTERPOLATION

Using the software product I developed, you can find the best curve to approximate a set of points with. Most of the points will then be positioned on a concave or convex contour or on a curve that has only one turning point. The outline will be made out of only one curve, passing through the two extreme points and the intermediate points (*see picture 5*). The main steps taken by the software are:

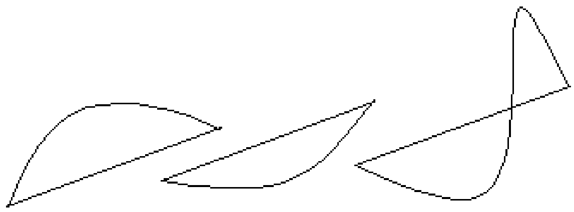


Fig. 4, The graphical form of the Bézier curve

S1. It draws an imaginary line between the extreme points. The set of points can be approximated with a Bézier curve only if the points are positioned above or below the line or if it has only one intersection point with the curve. Otherwise, the set of points will be divided in several sets, until this condition will be fulfilled.

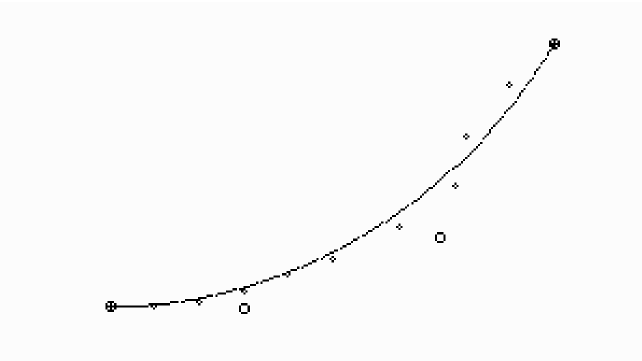


Fig 5, The curve, passing through the two extreme points and the intermediate points

S2. It calculates the area determined by the Bézier polynom, lines $y=y_i$ and $y=y_{i+1}$ and the Ox axis. The Bézier polynom can be obtained by joining with a line each intermediate point with the following one.

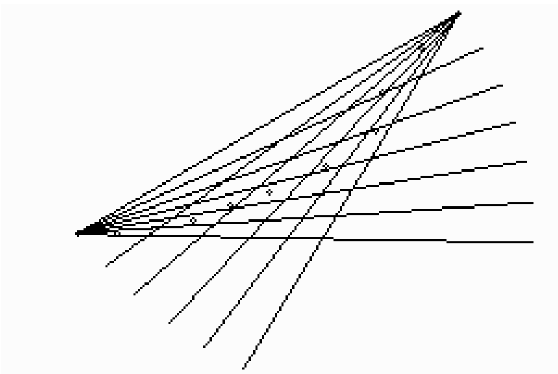


Fig. 6, It draws an imaginary line between nodes and extreme points

S3. It determines the inclination and the analytical expression of the tangents in the two nodes. In order to do this(*see picture 6*), the two segments will be rotated until all points fit between them.

S4. It establishes the variation domain of the control point. The point will be positioned in one of the two possibilities:

- if all of the points are on one side the cord, the control points will be positioned between the nodes and the intersection of the tangents.
- if the points intersect the cord, the control points will be positioned between the highest and the lowest point.

S5. It will determine *(see picture 7)*:

S5.1. The polynomial coefficients of the Bézier curves, calculated with relation (12).

S5.2. The set of points x(t) and y(t) that determine the Bézier curve, by giving various values to the t parameter, from within its domain: [0,1].

S5.3 The graphic form of the Bézier forms, by signalling the x(t) and y(t) points on the display, as they were calculated in the previous sequence.

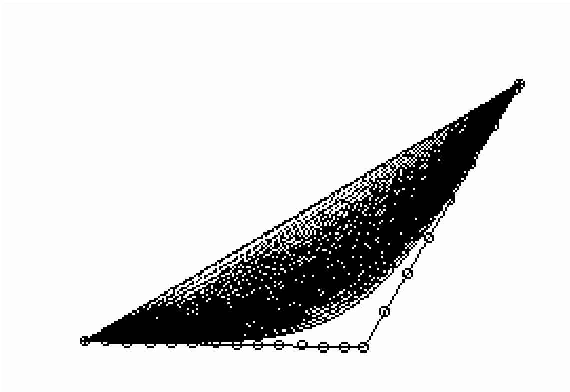


Fig. 7, Bézier curves by giving various to the control points

S5.4 The area determined by the polygon made out of the intermediate points, lines $y=y_i$ and $y=y_{i+1}$ and the Ox axis.

S5.5 The difference between the absolute value of the area determined in sequence S2 and the absolute value of the area determined in sequences

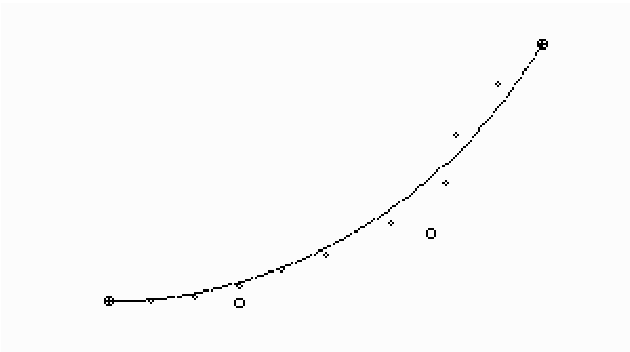


Fig. 8, It redraws the Bézier curve as it was chosen

S6. After ending the calculus cycle, it selects the minimum value of the difference between the area of the Bézier polygon and the Bézier curve, as determined in sequence S5.5.

S7. It redraws the Bézier curve as it was chosen in sequence S6, signalling the points that can be found on it from the initial set of points *(see picture 8)*.

This curve will be the Bézier curve that best approximates the set of points.

5. DETERMINING THE MODELING OF THE PATTERN SHAPES BELONGING TO A SHOE PRODUCT BY EXPERIMENTING WITH BÉZIER

The main working sequences in modeling complex outlines – more precisely, already-made patterns of footwear products – are as follows:

1. We mark the nodes on the outlines. These will mark those portions of the curve that can be approximated with a Bézier interpolating function.

2. Each portion will be digitally modeled. The coordinates of the points will be registered in a data file. *Picture 9* presents the way the nodes have been marked on the top-of-the-boot outline.

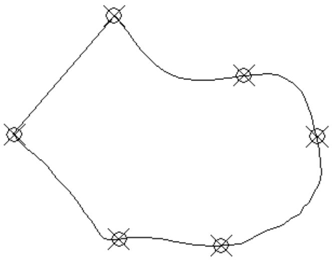


Fig. 9, The parts of the footwear products

3. An interactive session follows: by the means previously exposed we interpolate the points between two nodes. By executing the program, we can see the graphic results and also register in a data file the analytical forms of each interpolating function related to each curve portion.

Table nr. 1

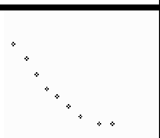
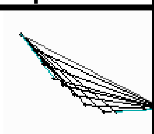
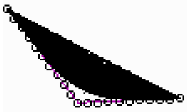
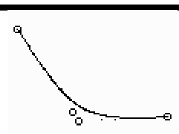
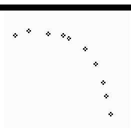
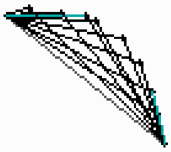
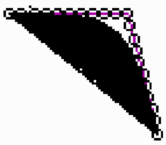
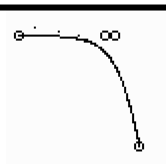
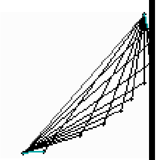
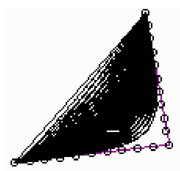
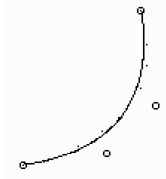
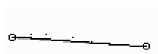
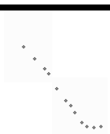
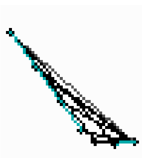
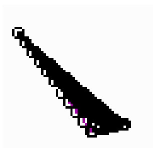
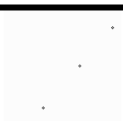
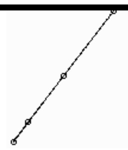
Discretized curve per portion	Looking for tangents at the group of points	Generating Beziér curves	The Bezier that best approximates the group of digitized points	The analytical form of the curve
				x(t)= $50.9t^3-56.7t^2+63.8t+110$ y(t)= $21.2t^3+78.5t^2+88.3t+146$
				x(t)= $27.4t^3-68.4\ t^2+76.9t+176$ y(t)= $-30\ t^3 + 115t$
				x(t)= $11.8t^3-76.0\ t^2+17.1t+212$ y(t)= $-3.5\ t^3+50.6\ t^2\ 103.1t+85$
				$x(t)=102\ t^3-153\ t^2 +165$ $y(t)=-8\ t^3+12\ t^2 +29$
				x(t)= $-22.5\ t^3+19.6t^2+2.0t+114$ y(t)= $22.6\ t^3+4.9\ t^2 -5.5t+33$
				x(t)= $-2.8\ t^3+31.2\ t^2\ \ 21.6t+60$ y(t)= $-3.\ t^3+35.9\ t^2+27.0t+86$

Table 1 shows the main sequences of the program, as it has to executed for the top-of-the-boot pattern:

– **Column 1** shows the number of points that will be approximated with a continuous curve.
– **Column 2** shows the sequence necessary in obtaining the tangents to the points' group.
– **Column 3** presents all Bézier curves that can be drawn between two nodes, using different variations of the control points on the tangent.
– **Column 4** contains, for each group of points, the Bézier curve that best approximates them.
– **Column 5** presents the analytical form of the curve portions that belong to the modeled pattern.

The information in the table is relevant for the rich opportunities offered by the program. One can easily notice how easy it becomes to model irregular shapes, belonging to concave, convex, concave-convex or to straight outlines.

After modeling each group of points, we register the graphical interpolated shape by determining the values x(t) and y(t) in points belonging to the interval [0,1] (see picture 10). In order to make more clear the working accuracy, on the modeled outline, we marked up the nodes and the tangents to the points' group as well as the corresponding control points.

6. CONCLUSIONS

1. By using a single function, Beziér polinoms allow the approximation of a group of points found on a concave, convex or concave-convex curve.
2. The patterns of the shoe products have irregular geometrical shapes, with plenty of concavities and convexities. This is why one must split them intro curve portions before modeling them.
3. The result of modeling a shoe pattern has many advantages because it can approximated only by using several interpolating polinoms, depending on their configuration. The polinoms will represent the numerical database of the pattern modified in CAD sessions.

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