PROPERTIES OF COEFFICIENTS OF ANALYTICAL PERIODIC FUNCTIONS

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ABSTRACT: If function f (x) is analytical, then it can be presented by convergent exponential sequence which, due to its convergence (d'Alembert's criterion), can be differentiated and integrated, member by member and as a rule it has Taylor's coefficients. However, in this work we have determined the properties of coefficient sequence when function is analytical, but also periodic with period ω. We have also shown that for the periodic function the coefficient sequence has the following form $a_k = \frac{\varphi^{(k)}(\omega)}{k!} = \frac{\varphi^{(k)}(2\omega)}{k!} = \frac{\varphi^{(k)}(n\omega)}{k!}$. In this way we have obtained infinite number of Taylor's formulae which are valid near the points 0, 2ω, nω.

INTRODUCTION.

Let us have the analytical periodic function:

$$f(x+\omega) = f(x) \tag{1}$$

Where the period $\omega = const$ and the points *x* and $x+\omega$ are within defined scope of function *f(x)*. From analycity

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots, \tag{2}$$

it also follows that:

$$f(x+\omega) = a_0 + a_1(x+\omega) + a_2(x+\omega)^2 + a_3(x+\omega)^3 + \ldots + a_n(x+\omega)^n + \ldots, \tag{3}$$

If in (3) we exponentiate binomials, then from (1), (2) and (3) follows the equality:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots = a_0 + a_1(x+\omega) + a_2(x+\omega)^2 + a_3(x+\omega)^3 + \ldots + a_n(x+\omega)^n + \ldots,$$

1. PRORERTIES OF COEFFICIENT SEQUENCE

After visible exclusions, it follows the identity:

$$\begin{aligned} & a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + \ldots + \\ & + x\left[2a_2\omega + 3a_3\omega^2 + 4a_4\omega^3 + \ldots\right] + \\ & + x^2\left[3a_3\omega + 6a_4\omega^2 + 10a_5\omega^3 + \ldots\right] + \\ & + x^3\left[4a_4\omega + 10a_5\omega^2 + 60a_6\omega^3 + \ldots\right] + \ldots \equiv 0 \end{aligned} \tag{4}$$

In order to make identity (4) valid for every *x* from the defined scope of function *f(x)*, the following should be valid:

$$\begin{cases} a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + \ldots = 0 \\ 2a_2\omega + 3a_3\omega^2 + 4a_4\omega^3 + \ldots = 0 \\ 3a_3\omega + 6a_4\omega^2 + 10a_5\omega^3 + \ldots = 0 \\ 4a_4\omega + 10a_5\omega^2 + 60a_6\omega^3 + \ldots = 0 \\ \vdots \end{cases} \tag{5}$$

Since $\omega \neq 0$, because in opposite case there is no periodicity, all equations (5) can be divided by ω , so there is

$$\begin{cases} a_1 + a_2\omega + a_3\omega^2 + a_4\omega^3 + ... = 0 \\ 2a_2 + 3a_3\omega + 4a_4\omega^2 + ... = 0 \\ 3a_3 + 6a_4\omega + 10a_5\omega^2 + ... = 0 \\ 4a_4 + 10a_5\omega + 60a_6\omega^2 + ... = 0 \\ \vdots \end{cases} \tag{6}$$

This means that each coefficient of the sequence (2) depends on every succeeding coefficients and exponents of the period ω.

$$\begin{cases} a_1 = -a_2\omega - a_3\omega^2 - a_4\omega^3 - ... = 0 \\ 2a_2 = -a_3\omega - 4a_4\omega^2 - ... = 0 \\ 3a_3 = -6a_4\omega - 10a_5\omega^2 - ... = 0 \\ 4a_4 = -10a_5\omega - 60a_6\omega^2 - ... = 0 \\ \vdots \end{cases} \tag{6'}$$

which is a useful fact, regardless of not being able to find immediately the coefficients of periodic and analytical function f(x).
Two already considered, successive identities points out to the very important characteristic i.e. connection between periodicity and derivative.
From (2), for $x = \omega$ it follows that:

$$f(\omega) = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + ...$$

i.e.

$$f(\omega) - a_0 = a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + ... \tag{7}$$

Considering the equation (5), it follows that $f(\omega) - a_0 = 0$ or

$$a_0 = f(\omega) \tag{8}$$

Theorem 1. In analytical periodic function the first coefficient in its sequence (2), i.e. zero exponent, is the value of the periodic function in the point $x = \omega$ of one full period.
Based on sequence theory, analytical function can be differentiated, member by member and its derivative is also analytical function and derivative sequence has the same radius of convergence. Therefore, from (2), it follows that:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ... + na_nx^{n-1} + ...,$$

And for $x = \omega$ it follows that

$$f'(\omega) = a_1 + 2a_2\omega + 3a_3\omega^2 + 4a_4\omega^3 + ...$$

From this it follows that:

$$f'(\omega) - a_1 = 2a_2\omega + 3a_3\omega^2 + 4a_4\omega^3 + ...$$

and the right side is the second equation in the system (5). It follows that $a_1 = f'(\omega)$ i.e.

$$a_1 = \frac{f'(\omega)}{1!}$$

Therefore the coefficient a_1 in the sequence (2) for periodic analytical function is equal to the first derivative of that function in the point ω.
In the same way it is

$$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \ldots$$
$$f''(\omega) - 2\cdot 1\cdot a_2 = 2(3a_3\omega + 6a_4\omega^2 + \ldots) or$$
$$\frac{f''(\omega) - 2\cdot 1\cdot a_2}{2} = 3a_3\omega + 6a_4\omega^2 + 10a_5\omega^3 \ldots$$

and according to the third equation from (5) we obtain

$$\frac{f''(\omega) - 2\cdot 1\cdot a_2}{2} = 0 \quad \text{or} \quad a_2 = \frac{f''(\omega)}{2!}$$

We continue the procedure further for $f'''(x), f^{(4)}(x), \ldots$ and come to the induction premise

$$a_n = \frac{f^{(n)}(\omega)}{n!}. \tag{9}$$

Then, from the derivative

$$f^{(n+1)}(x) = (n+1)! a_{n+1} + (n+2)(n+1)! a_{n+2} x + \ldots,$$

by introducing $x = \omega$ and considering the identity of (5) we obtain the following

$$f^{(n+1)}(\omega) - (n+1)! a_{n+1} = 0 \quad \text{i.e.} \quad a_{n+1} = \frac{f^{(n+1)}(\omega)}{(n+1)!}$$

According to the well known Taylor's formula and Taylor's sequence for analytical function (according to which the kth member of the sequence depends on the derivative $f^{(k)}(x_0)$ in one fixed point x_0, except that this point now is the very period ω), this is something that could be expected. It means that this proof is the variant of classic Taylor's theorem, but for analytical and periodic functions.

Theorem 2. Coefficients of the exponential sequence of analytical periodic function with period $\omega = const$ have the following form $a_k = \frac{f^{(k)}(\omega)}{k!}$ where $f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k$.

Now, regarding the periodicity of $f(x) = f(x+\omega) = f(x+2\omega) = f(x+3\omega) = \ldots = f(x+n\omega)$, which is valid for every x, including x=0, we obtain the following:

$$f(0) = f(\omega) = f(2\omega) = f(3\omega) = \ldots = f(n\omega).$$

However, by differentiating the relation (1) we obtain that the functions $f'(x), f''(x), \ldots f^{(n)}x, \ldots,$ are also periodic with period ω, so it follows that the same is valid for them as for the function f(x).

$$\begin{cases} f'(0) = f'(\omega) = f'(2\omega) = \ldots = f'(n\omega) \\ f''(0) = f''(\omega) = f''(2\omega) = \ldots = f''(n\omega) \\ \vdots \\ f^{(n)}(0) = f^{(n)}(\omega) = f^{(n)}(2\omega) = \ldots = f^{(n)}(n\omega) \end{cases} \tag{10}$$

In this way we obtain another possible formula

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(n\cdot\omega)}{k!} \cdot x^k, \quad n=0,1,2,\ldots, \tag{11}$$

which can be used successfully, instead in ε (є) environment $|x - x_0| < \varepsilon$ of every separate point x_0, in some ω-environment of the same point $x_0 : x = x_0 + n\cdot\omega$.

However, the above regular repetition of derivatives shows that the certain number of coefficients in (11) will always repeat.

Therefore, let n be fixed and n<k, then the relation (11) can be written in the following form

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(n\omega)}{n!}\cdot\frac{x^k}{(n+1)(n+2)........k}. \tag{12}$$

Since $f^{(k)}(n\omega)=f^{(k)}(0)$, it follows that

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{n!}\cdot\frac{x^k}{(n+1)(n+2)........k}. \tag{13}$$

If n is fixed number, then, since every derivative including the k^{th} derivative of periodic function is periodic function, it can be repeated. When it is going to be repeated, it depends on properties of function f(x). Since we are interested in periodic functions, which have differential equations as solutions, we are going to try to connect this fixed n with sequence n of differential equation. [1] Therefore, in the sum (12), if k has values 0,1,2,…(n-1), then the values of the derivative

$$f^{(0)}(0), f'(0), f''(0),..., f^{(n-1)}(0) \tag{14}$$

are different, while $f^{(n)}(0)$ repeats: $f^{(n)}(0)=f^{(0)}(0)=y_0$. If there were no repeating, then solution $y(x)=f(x)=f^{(0)}(x)$ would not be periodic. In this way its derivatives would not be periodic either.

Theorem 3. Let us have analytical differential equation of the n^{th} order in normal form, which does not contain argument x

$$y^{(n)}=F(y,y',y'',....,y^{(n-1)})$$

where F is analytical in comparison to all its arguments, which has periodic solution $y=f(x),\ \ f(x+\omega)=f(x)$, then the final sequence (14) can contain different derivatives, but the n^{th} derivative keeps regularly repeating in point ω.

Let us have

$$y^{(n)}(x+\omega)=F(y(x+\omega),y'(x+\omega),...,y^{(n-1)}(x+\omega))$$

Since the derivatives of analytical periodic function are also periodic function with the same period it follows that

$$y^{(n)}(x+\omega)=F(y,y',...,y^{(n-1)})$$

If we put here x=0, we obtain the following:

$$y^{(n)}(\omega)=F(f(0),f'(0),...,f^{(n-1)}(0))=y^{(n)}(0),$$

i.e. the n^{th} derivative repeats.

There is one very important special case when linear differential equation of the n^{th} order has the following form:

$$y^{(n)}=a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+...+a_n(x)y,$$

where $a_i(x)$ or periodic functions have the same period or constants. Let us have following in the point $x+\omega$:

$$y^{(n)}(x+\omega)=a_1(x+\omega)y^{(n-1)}(x+\omega)+...+a_n(x+\omega)y(x+\omega),$$

Since the coefficients and derivatives till the $(n-1)^{th}$ order are periodic, it follows that

$$y^{(n)}(x+\omega)=a_1(x)y^{(n-1)}(x)+a_2(x)y^{(n-2)}(x)+...+a_n(x)y(x),$$

and if we put x=0, we obtain

$$y^{(n)}(\omega)=a_1(0)y^{(n-1)}(0)+a_2(0)y^{(n-2)}(0)+...+a_n(0)y(0)=y^{(n)}(0),$$

i.e. the n^{th} derivative regularly repeats.

Theorem 4. If the normal linear homogenous differential equation of the n^{th} order, the coefficients of which are periodic functions with the same period has the periodic solution y=f(x), then its derivatives till the (n-1)th derivative can be different, but the nth derivative regularly repeats. [2]

In connection with the analytical presentation (13) the following is valid

Theorem 5. If f(x) is periodic analytical solution of some analytical differential equation of the nth order, then the coefficients $\frac{f^{(k)}(0)}{n!}$ for k=0,1,2,…,(n-1) are different, but starting from n there are repetitions, while the factor $\frac{x^k}{(n+1)(n+2)(n+3).......k}$ constantly changes with the increase of k.

Therefore, the sequence of analytical and periodic function repeats its one factor.

Theorem 6. The sequence of analytical and periodic solution of the Hill's equation $y'' = \Phi(x) \cdot y$, $\Phi(x)$- periodic function, repeats part of each its addend at interval of two members (steps).

Proof: Hill's equation can have periodic solution, but it does not have to. Let us suppose that it has periodic solution and let it be p(x). In this case $p'(x)$ is also periodic function and it follows:

$$p''(x+\omega) = \Phi(x+\omega) \cdot p(x+\omega)$$

Based on premise that $\Phi(x)$ and p(x) are periodic functions, it follows that:

$$p''(x+\omega) = \Phi(x) \cdot p(x)$$

In x=0 it is as follows

$$p''(\omega) = \Phi(0) \cdot p(0) = p''(0)$$

Therefore, $p''(\omega)$ repeats after $p'(0)$ and p(0) and further after every two steps.

Example: Let us have equation of harmonious oscillations $y'' = -y$, the solutions of which are *sin x* and *cos x*. From the equation, it follows:

$$\begin{aligned}
&y''' = -y' \\
&y^{(4)} = -y'' = -(-y) = y \\
&y^{(5)} = -y''' = -(-y') = y' \\
&y^{(6)} = -y^{(4)} = -y \\
&\vdots
\end{aligned}$$

Based on this, we can see that all derivatives depend only either on *y* or y', therefore on two values and two is also order of this equation. In the period of solution, i.e. in the point $x = \omega$, owing to the relation (10), the values of the derivative are either $\pm y(0)$ or $\pm y'(0)$. So if we write sequence (13) for both solutions, we obtain

$$\begin{aligned}
f(x) = &\frac{y_0}{0!} \cdot \frac{x^0}{0!} + \frac{y'(0)}{0!} \cdot \frac{x^1}{(0+1)} + \frac{-y_0}{1!} \cdot \frac{x^2}{(1+1)} + \frac{-y'(0)}{2!} \cdot \frac{x^3}{(2+1)} + \\
&+ \frac{y_0}{2!} \cdot \frac{x^4}{(2+1)(2+2)} + \frac{y'(0)}{2!} \cdot \frac{x^5}{(2+1)(2+2)(2+3} + \\
&+ \frac{-y_0}{2!} \cdot \frac{x^6}{(2+1)(2+2)(2+3)(2+4)} + \\
&+ \frac{-y'(0)}{2!} \cdot \frac{x^7}{(2+1)(2+2)(2+3)(2+4)(2+5)} + \ldots
\end{aligned}$$

If we group members with y_0 and y_0' we obtain

$$f(x) = y_0\left[1 - \frac{1}{2!}x^2 + \frac{1}{2!}\frac{x^4}{3\cdot 4} - \frac{1}{2!}\frac{x^6}{3\cdot 4\cdot 5\cdot 6} + \dots\right]$$
$$+ y_0'\left[1 - \frac{1}{2!}\frac{x^3}{3} + \frac{1}{2!}\frac{x^5}{3\cdot 4\cdot 5} - \frac{1}{2!}\frac{x^7}{3\cdot 4\cdot 5\cdot 6\cdot 7} + \dots\right]$$

This is characteristic (and familiar) repetition of the derivative $\frac{y^{(k)}(0)}{2!} = \frac{(\pm 1)\vee 0}{2!}$ for *k=2,3,4,…*

CONCLUSION

These theorems could be useful for recognizing which exponential order defines periodic functions and which not. It is necessary to define even more strict criteria. Based on aforesaid, we can conclude that when the function *f(x)* is analytic and periodic with period ω, then the coefficients of the sequence (2), which represent it, have the following form:

$$a_0 = f(\omega) = f(2\omega) = \dots = f(n\omega)$$
$$a_1 = \frac{f'(\omega)}{1!} = \frac{f'(2\omega)}{1!} = \dots = \frac{f'(n\omega)}{1!}$$
$$a_2 = \frac{f''(\omega)}{2!} = \frac{f''(2\omega)}{2!} = \dots = \frac{f''(n\omega)}{2!}$$
$$\vdots$$
$$a_k = \frac{f^{(k)}(\omega)}{k!} = \frac{f^{(k)}(2\omega)}{k!} = \dots = \frac{f^{(k)}(n\omega)}{k!}$$

In this way we obtain infinite number of Talyor's formulae which are valid near the points $0, \omega, 2\omega, \dots, n\omega$.

REFERENCES:

[1] E. Kamke: Handbook of Ordinary Differential Equations, Science, Moscow, 1971; (in Russian)
[2] Cesari: Asymptotic behavior and stability of solutions of ordinary differential equations, World, Moscow: 1964. (in Russian)