ITERATION SEQUENCES METHOD FOR THE COMPLETE LINEAR HOMOGENOUS DIFFERENTIAL EQUATION OF THE SECOND ORDER ITERATIONS. EVALUATION.

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Abstract: Iterations as a method applied for solving differential equations are valid for many classes of equations, because they do not require the existence of all derivatives, but just continuity and Lipschitz's condition. The first trial has produced excellent results, because by applying this method, we have successfully solved the equation of the linear oscillations with non-constant period $y'' + a(x) \cdot y = 0$, a(x)>0. In this work we shall solve the complete linear homogenous equation of the second order $y'' = a(x) \cdot y' + b(x) \cdot y$, which represents equation of oscillations resistances. We shall also prove that the integral operator is contracting and establish two new theorems.

INTRODUCTION. Let us have the complete linear homogenous differential equation in its normal form:

(1) $$y'' = a(x) \cdot y' + b(x) \cdot y$$

If in the equation (1) the coefficients *a(x)* and *b(x)* are continuous, then the equation has a unique twice continuous and differentiable solution *y(x)*, which can be determined by iterations. The same is determined by finding the first integrals from the normal form (1) of the given equation.

(2) $$y' = C_1 + \int_0^x a(x) \cdot y' dx + \int_0^x b(x) \cdot y dx$$

and further:

(3) $$y = C_1 \cdot x + C_2 + \int_0^x \int_0^x a(x) \cdot y' \cdot dx^2 + \int_0^x \int_0^x b(x) \cdot y \cdot dx^2$$

It is clear that the equation (1) and the integral equation (3) are equivalent in terms of solving. However, if we define the direct iterations by relation (3), then in members of iteration sequence $\{y^{[n]}(x)\}$, y' would be replaced with formulation (2). This is not the most convenient method for direct iterations, because the practice shows that frequent mistakes are made in iteration steps due to irregular and even unnecessary double or multiple substitution of y' with *y*. That is why we transform the relation (2). We take the first integral in (2) and by applying the following:

$$u = a(x), \; dv = y' \cdot dx$$

we make the partial integration:

$$\int_0^x a(x) \cdot y' dx = a(x) \cdot y - \int_0^x a'(x) \cdot y \cdot dx$$

In that way we obtain:

$$y' = a(x)\cdot y - \int_0^x a'(x)\cdot y\cdot dx + \int_0^x b(x)\cdot y\cdot dx + C_1$$

$$= a(x)\cdot y + \int_0^x [b(x) - a'(x)]\cdot y\cdot dx + C_1$$

It follows that:

(4) $$y = \int_0^x a(x)\cdot y\cdot dx + \int_0^x\int_0^x [b(x) - a'(x)]\cdot y\cdot dx^2 + C_1\cdot x + C_2 ,$$

where C_1 and C_2 are integration constants.

The integral form (4) of the equation (1) contains one single and one double integral of the searched function *y(x)*. The form (4) might be better from the subjective point of view than the direct iteration, because it leads to three instead of four quadratures. The same number is contained by the first integral (3). The problem in the integral equation (4) is the appearance of the derivative $a'(x)$ of the coefficient *a(x)* for which we supposed that it is only continuous function. [2] That is why the form (4) is not convenient for more general theoretical proofs, because it requires differentiability of the coefficient *a(x)*, which is unnecessarily high requirement. Consequently, we shall do the theoretical part with (3), while the integral form (4) will be used in practice. In that sense, we shall introduce the new unknown function *Z=Z(x)* into the equation (1), by substitution $y' = Z$. In this way the problem is turned into the system of two plain differential equations of the first order:

(5) $$\begin{cases} y' = Z \\ Z' = a(x)\cdot Z + b(x)\cdot y \end{cases}$$

The system (5) is equivalent to the system containing two integral equations in which integration constants C_1 and C_2 and single integrals are present. [6]

(6) $$\begin{cases} y = \int_0^x Z(x)dx + C_1 \\ Z = \int_0^x (a(x)\cdot Z + b(x)\cdot y)\cdot dx + C_2 \end{cases}$$

By integral equations (6) we define the iterations by using the naturally suggested relations:

(7) $$\begin{cases} y^{[n]} = \int_0^x Z^{[n-1]}(x)dx + C_1, \\ Z^{[n]} = \int_0^x (a(x)\cdot Z^{[n-1]}(x) + b(x)\cdot y^{[n-1]}(x))\cdot dx + C_2, n = 1,2,3... \end{cases}$$

1. ESTABLISHING TWO NEW THEOREMS

Without reducing the general nature, we shall suppose that for the initial approximations $y^{[0]}$ and $Z^{[0]}$ the following is valid: $y^{[0]}(x) = y(0) = y_0$ and $Z^{[0]}(x) = Z(0) = Z_0$.

From (7), it obviously follows that:

$y^{[1]}(0) = C_1$ and $Z^{[1]}(0) = C_2$.

It also follows that

(8) $y(0) = C_1$ and $y'(0) = Z(0) = C_2$

For *n=1* we obtain the first members of the iterations sequence:

$$y^{[1]} = \int_o^x Z^{[0]}(x)dx + C_1 = Z(0)\cdot x + C_1 = C_2\cdot x + C_1$$

$$Z^{[1]}=\int_o^x(a(x)Z^{[0]}(x)+b(x)y^{[0]}(x))dx+C_2=C_2\int_0^x a(x)dx+C_1\int_0^x b(x)dx+C_2$$

based on which there follows an interesting theorem which has not been formulated up to now.

Theorem 1. *The first iteration of the unknown function y(x) of the linear homogenous differential equation of the second order is the linear function, i.e. the tangent:*

$$y=C_2\cdot x+C_1=y'(0)\cdot x+y(0)$$

while the first iteration of the back up function Z(x) is the linear combination of integrals of the coefficients of the differential equation. [4]

This statement is not without theoretical meaning, because, now, we can evaluate the first iterations, which can be the basic means of analysis:

$$\left|y^{[1]}(x)\right|<\left|C_2\right|\cdot x+\left|C_1\right|,$$

$$\left|Z^{[1]}(x)\right|<C_2\cdot\int_0^x\left|a(x)\right|dx+C_1\int_0^x\left|b(x)\right|dx+C_2$$

Since the coefficients *a(x)* and *b(x)* are continuous, they are limited functions, so there are positive real constants *A* and *B,* where

$\left|a(x)\right|<A$ and $\left|b(x)\right|<B$.

Thus, the final evaluation of the first iterations should be:

$$y^{[1]}<C_2\cdot x$$

$$Z^{[1]}<(AC_2+BC_1)\cdot x+C_2<(AC_2+BC_1)\cdot x$$

The next step includes the members of iteration sequence which we obtain for *n*=2.

$$y^{[2]}=\int_0^x Z^{[1]}(x)dx+C_1=\int_0^x[C_2\int_0^x a(x)dx+C_1\int_0^x b(x)dx]+C_2=$$

$$=C_1\int_0^x\int_0^x b(x)dx^2+C_2(1+\int_0^x\int_0^x a(x)dx^2,$$

$$Z^{[2]}=\int_0^x[a(x)Z^{[1]}(x)dx+b(x)y^{[1]}(x)]dx+C_2=$$

$$=\int_0^x\{a(x)[C_2\int_0^x a(x)dx+C_1\int_0^x b(x)dx+C_2]+b(x)(C_2\cdot x+C_1)\}dx+C_2=$$

$$=C_2(1+\int_0^x xb(x)dx+\int_0^x a(x)dx+\int_0^x\int_0^x a(x)dx^2+$$

$$+C_1(\int_0^x b(x)dx+\int_0^x a(x)dx\int_0^x b(x)dx)$$

Consequently, it follows that:

$$\left|y^{[2]}\right|<C_1B\int_0^x\int_0^x dx^2+C_2(1+A\int_0^x\int_0^x dx^2)=$$

$$=C_1B\cdot\frac{x^2}{2!}+C_2(1+A\frac{x^2}{2!})=P_2(x;A,B,C_1,C_2)$$

and

$$\left|Z^{[2]}\right|<C_2(1+B\int_0^x xdx+A\int_0^x dx+A\int_0^x\int_0^x dx^2)+C_1(B\int_0^x dx+AB\int_0^x\int_0^x dx^2)$$

$$=C_2\cdot(1+Ax+B\cdot\frac{x^2}{2!}+A^2\frac{x^2}{2!})+C_2(B\cdot x+AB\frac{x^2}{2!})=$$

$$=Q_2(x;A,B,C_1,C_2)$$

P_2 and Q_2 are polynomials of the second degree. Its highest members are $(C_1B + C_2A)\cdot\frac{x^2}{2!}$ and $[C_2(B + A^2) + C_1AB]\cdot\frac{x^2}{2!}$ which will be important for the asymptotics.

Further for *n=3,* we obtain:

$$y^{[3]} = \int_0^x Z^{[2]}dx + C_1 = \int_0^x [C_2(1 + \int_0^x xb(x)dx + \int_0^x a(x)dx + \int_0^x\int_0^x a(x)dx^2] +$$

$$+ C_1(\int_0^x b(x)dx + \int_0^x a(x)dx\int_0^x b(x)dx)]dx + C_1 =$$

$$= C_2(1 + \int_0^x\int_0^x xb(x)dx^2 + \int_0^x\int_0^x a(x)dx^2 + \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2) +$$

$$+ C_1(x + \int_0^x\int_0^x b(x)dx^2 + \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x b(x)dx^2$$

This leads to the following evaluation:

$$\left|y^{[3]}\right| < C_2(1 + B\frac{x^2}{2!} + A\frac{x^2}{2!} + A^2\frac{x^3}{3!}) + C_1(x + B\frac{x^2}{2!} + AB\frac{x^3}{3!}) =$$

$$= P_3(x; A, B, C_1, C_2)$$

where P_3 is the polynomial of the third degree and its highest member has the following form $\frac{x^3}{3!}(A^2\cdot C_2 + A\cdot B\cdot C_1)$.

There is a complete analogy with:

$$Z^{[3]} = \int_0^x (a(x)Z^{[2]}(x) + b(x)y^{[2]}(x))dx + C_2 =$$

$$= \int_0^x \left\{ a(x)[C_2(1 + \int_0^x xb(x) + \int_0^x a(x)dx + \int_0^x\int_0^x a(x)dx^2) + \right.$$

$$\left. + C_1(\int_0^x b(x)dx + \int_0^x a(x)dx\int_0^x b(x)dx)] + b(x)(C_1\int_0^x\int_0^x b(x)dx^2 + C_2(1 + \int_0^x\int_0^x a(x)dx^2)) \right\} dx + C_2 =$$

$$= C_2(1 + \int_0^x a(x)dx + \int_0^x a(x)dx\int_0^x xb(x)dx + \int_0^x\int_0^x a(x)dx^2 + \int_0^x a(x)dx\int_0^x\int_0^x a(x)dx^2 +$$

$$+ \int_0^x b(x)dx + \int_0^x b(x)dx\int_0^x\int_0^x a(x)dx^2) +$$

$$+ C_1(\int_0^x\int_0^x b(x)dx^2 + \int_0^x\int_0^x a(x)dx^2\int_0^x b(x)dx^2 + \int_0^x b(x)dx^2\int_0^x\int_0^x b(x)dx^2)$$

Consequently there follows the evaluation:

$$\left|Z^{[3]}\right| < C_2(1 + A\cdot x + AB\frac{x^3}{3!} + A^2\frac{x^2}{2!} + A^3\frac{x^3}{3!} + Bx + AB\frac{x^3}{3!}) +$$

$$+ C_1(AB\frac{x^2}{2!} + A^2B\frac{x^3}{3!} + B^2\frac{x^3}{3!}) = Q_3(x; A, B, C_1, C_2)$$

where Q is a polynomial of the third degree and its highest member is $[C_2(A^3+2A\cdot B)+C_1(B^2+A^2B)]\cdot\frac{x^3}{3!}$.

Referring to evaluations of successive approximations for continuous coefficients *a(x)* and *b(x)*, the previous would be enough for one total induction. Therefore:

$$y^{[1]}<C_2\cdot x \qquad Z^{[1]}<(AC_2+BC_1)\cdot x$$

$$y^{[2]}<(C_1B+C_2A)\cdot\frac{x^2}{2!} \qquad Z^{[2]}<(C_2(B+A^2)+C_1AB))\cdot\frac{x^2}{2!}$$

$$y^{[3]}<(A^2C_2+ABC_1)\cdot\frac{x^3}{3!} \qquad Z^{[3]}<[C_2(A^3+AB)+C_1(B^2+A^2B)]\cdot\frac{x^3}{3!}$$

$$\vdots \qquad \vdots$$

The theorem of the polynomials says that in the polynomials of the n^{th} degree dismissing members of the exponent less than n can be regulated with $K\cdot x^n$ (*K* - constant). Thence, it can be concluded that for all approximations the following is valid:

(9) $$y^{[n]}(x);Z^{[n]}(x)<K\cdot\max(C_1C_2)\cdot(A+B)^n\cdot\frac{x^n}{n!}$$

This is very important inductive evaluation for the system of equations (3), but also for the systems which occur during solving the equations of the n^{th} order $L(y)=0$ with coefficients $a_0,a_1,a_2,...,a_n$ on a definite interval *[0,x]*.

Theorem 2. *For the evaluation of the iterations of the n^{th} step and for the solution y(x) and back up function Z(x), in linear homogenous differential equation of the second order (1), the evaluations (9) are valid, where K is the constant which depends on the iteration sequence, while* $A=\max|a(x)|$, $B=\max|b(x)|$ *and C_1 and C_2 are integration constants or initial values.*

We are going to use the theorem 2 in order to prove the most important thing when talking about iteration, i.e. the convergence of the iteration sequence. The sequence $\{y^{[n]}(x)\}$ converges toward some limit function *y(x)* only if the integral operator in the system (7) is contracting. In order to prove contraction of the operator, we will use the inductive evaluation (9) based on which:

$$\left|y^{[n]}-y^{[n-1]}\right|<\left|KM(A+B)^n\cdot\frac{x^n}{n!}-KM(A+B)^{n-1}\cdot\frac{x^{n-1}}{(n-1)!}\right|<$$

$$<\left|KM(A+B)^{n-1}\cdot\frac{x^{n-1}}{(n-1)!}\right|\cdot\left|\frac{x}{n}-1\right|<KM\frac{|(A+B)\cdot x|^{n-1}}{(n-1)!} \text{ i.e.}$$

(10) $$\left|y^{[n]}-y^{[n-1]}\right|<KM\frac{|(A+B)\cdot x|^{n-1}}{(n-1)!}$$, *n=1,2,3…*

The sequence $KM\cdot\sum_{n=1}^{\infty}\frac{[(A+B)\cdot x]^{n-1}}{(n-1)!}$ is convergent according D'Alambert's criterion in every definite interval *[0,x]*. Thus, its general member tends toward zero, i.e. $A_n=\frac{K\cdot M[(A+B)\cdot x]^{n-1}}{(n-1)!}\to 0,\ n\to\infty$.

Let us formulate now functional sequence:

(11) $y^{[0]}(x)+\sum_{n=1}^{\infty}\left[y^{[n]}(x)-y^{[n-1]}(x)\right]$

The relation (10) shows that the sequence (11) is absolutely and uniformly convergent according to Weierstrass' criterion, because it has been majorated with convergent potential sequence, thus making the sequence of partial sums of this sequence also convergent. [1] However the nth partial sum of the sequence (11) is exactly n-approximation of $y^{[x]}$, because

$$y^{[0]}+\left[y^{[1]}-y^{[0]}\right]+\left[y^{[2]}-y^{[1]}\right]+\left[y^{[3]}-y^{[2]}\right]+...+\left[y^{[n]}-y^{[n-1]}\right]=y^{[n]}(x)$$

Therefore the sequence (11) and the iteration sequence $\left\{y^{[n]}(x)\right\}$, as a sequence of partial sums converge toward the same limit function $y^*(x)$, i.e.

$$\lim_{n\to\infty} y^{[n]}(x)=y^*(x)$$

In the completely analogical way, we also prove that $\lim_{n\to\infty} Z^{[n]}(x)=Z^*(x)$, because the inductive evaluation (9) is also valid for iteration sequence $\left\{Z^{[n]}(x)\right\}$

Therefore, the sequences (7) of the successive approximations are convergent, so there are limit values. [3]

$$\lim_{n\to\infty} y^{[n]}=\lim_{n\to\infty}\int_0^x Z^{[n-1]}(x)\cdot dx+C_1 \text{ and}$$

$$\lim_{n\to\infty} Z^{[n]}=\lim_{n\to\infty}\int_0^x \left[a(x)Z^{[n-1]}(x)+b(x)y^{[n-1]}(x)\right]\cdot dx+C_2$$

In this case, according to Laplace's rule, the limit value of the integral is equal to integral of the limit value. Therefore:

$$\lim_{n\to\infty} y^{[n]}=\lim_{n\to\infty}\int_0^x Z^{[n-1]}(x)\cdot dx+C_1=\int_0^x \lim_{n\to\infty} Z^{[n-1]}(x)\cdot dx+C_1=\int_0^x Z^*(x)dx+C_1,$$

$$\lim_{n\to\infty} Z^{[n]}=\lim_{n\to\infty}\int_0^x \left[a(x)Z^{[n-1]}(x)+b(x)y^{[n-1]}(x)\right]\cdot dx+C_2=$$

$$=\int_0^x \left[a(x)\lim_{n\to\infty} Z^{[n]}+b(x)\lim_{n\to\infty} y^{[n-1]}(x)\right]dx+C_2=$$

$$=\int_0^x \left[a(x)Z^*(x)+b(x)y^*(x)\right]dx+C_2$$

Therefore, we obtain:

$$y^*(x)=\int_0^x Z^*(x)dx+C_1$$

$$Z^*(x)=\int_0^x \left[a(x)Z^*(x)+b(x)y^*(x)\right]dx+C_2$$

The absolute and uniform convergence leads to the continuity and differentiability of the limit functions, so the last formulae can be differentiated:

$$(y^*)'=Z(x),$$

$$(Z^*)'=a(x)\cdot Z^*+b(x)\cdot y^*=$$

$$=a(x)\cdot(y^*)'+b(x)\cdot y^*=\left[(y^*)'\right]=(y^*)''$$

There follows that:

$$(y^*)'' = a(x)\cdot(y^*)' + b(x)y^*$$

It means that $y^*(x)$ is the solution of the given differential equation (1). This leads to:

Theorem 2. *The limit function $y^*(x)$ of sequence of the successive approximations (7) is the solution of the complete linear homogenous differential equation of the second order (1).*

The proof for this theorem has been derived in somewhat different way from the Picard-Lindelöf's method, because we have put the emphasis on elementary principle of contraction and derived the evaluations for the members of the iterations sequence. [5]

The uniqueness of the solution for the given initial conditions is proved by classical procedure.

CONCLUSION.

The equation $y'' = a(x)\cdot y' + b(x)\cdot y$ is an important differential equation especially in technical application, because under certain conditions it has oscillating solutions. Of course, everything depends on discriminant $D = b(x) - \frac{1}{2}a'(x) - \frac{1}{4}a^2(x)$ and its sign. If *D<0*, the equation has monotonous solutions in the form of hyperbolic functions:

$ch_{(a,b)}x$ and $sh_{(a,b)}x$

which can allow for certain non-monotony on a definite sub-interval, depending on the elementary non-monotony of the coefficients *a(x)* and *b(x)*. However, in the same time they secure the asymptotic monotony on the semi axis $[x_o, +\infty]$. In this case we obtain the equation of the hyperbolic type.

If *D>0*, the equation has oscillating solutions in the form of generalized trigonometric functions:

$\sin_{(a,b)} x$ and $\cos_{(a,b)} x$

In this case the equation is of elliptic type.

It is the trivial case when *D=0*, when the equation is of the parabolic type and has monotonous solutions which are the product of exponential and linear function:

$$y = e^{-\frac{1}{2}\int a(x)dx}\cdot(C_1\cdot x + C_2),$$

where C_1 and C_2 are integration constants.

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