MATHEMATICAL MODELING OF THE AIR FLOW INSIDE THE ATMOSPHERIC BOUNDARY LAYER

DEGERATU Mircea[1]; Liviu HAŞEGAN[1]; Nicolae Ioan ALBOIU[1]; Georgeta BANDOC[2]
[1]Technical University of Civil Engineering Bucharest,
e-mail: mircead@hidraulica.utcb.ro, nalboiu@hidraulica.utcb.ro;
[2]University of Bucharest, e-mail: bandoc@geo.unibuc.ro

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Abstract: The atmospheric boundary layer represents the lower part of the atmosphere where the air movement is influenced by its interaction with the Earth's surface. Inside, due to the formed vortices, the air flow has a turbulent character. The boundary layer thickness depends on the atmospheric wind velocity, the vertical layering of the temperature and on the geometrical characteristics of Earth's surface. Based on the specific methods used in fluid mechanics and the vortices discontinuity theory, the authors of the present paper achieved a study on the mathematical modeling of the air flow inside the atmospheric boundary layer.

1. INTRODUCTION

The atmospheric boundary layer (ABL) represents that part of the inferior planetary layer of the atmosphere wherein the air movements are strongly influenced by the interaction with the surface of the Earth, and its thickness depends on the value of wind velocity in the free atmosphere, on vertical temperature stratification and on the geometrical characteristics of the earth surface roughness (irregularity).

The air movements in the atmospheric boundary layer have, most often, a strong turbulent character as a result of vortex (eddies, air whirls) surfaces.

The study of turbulent flow in the atmospheric boundary layer, wherein average values are used to characterise the fields of characteristic parameters, is founded on the vortex discontinuity theory, theories which introduce the notion of vortex value defined by the mean of spatial correlation coefficient.

2. TURBULENT AVERAGE MOVEMENT EQUATIONS IN THE ATMOSPHERIC BOUNDARY LAYER

The atmospheric boundary layer equations which are about to be written bellow, will refer to a model in the following hypotheses: the current in area ABL is considered as having a neutral stratification, a valid hypothesis for a wind which is strong enough, and the air is considered an incompressible fluid.

The movement of the atmospheric air is described by the Reynolds equations (1) and the continuity equation (2). These equations have to be completed with phenomenological relations in order to close the equation system.

Starting with the turbulent average movement equations (Reynolds equations) and the continuity equation averaged by time and by eliminating the negligible terms (following an analysis of their size grade), on the basis of physical reasons, the equations which are describing the average movement in the atmospheric boundary layer are driven.

The orthogonal (cartesian) system of coordinates was chosen in such a way that x axis concur with the direction of shear stress on the earth surface, marked as τ_0 (Figure. 1). Thus, the x axis makes with the isobar an angle α_0. The y axis is normal on x axis, and both are on a horizontal plan and z axis is vertical.

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}=fV-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_x}{\partial z}$$

$$U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}=-fU-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_y}{\partial z} \qquad (1)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z}+g=0$$

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0 \qquad (2)$$

where U , V , W are the components of the average velocity on x , y axis and respectively z , f is Coriolis parameter, ρ is air density, p mean pressure, and τ_x and τ_y are the shear stresses following x and, respectively y axis, in the horizontal plan.

The shear stresses τ_x and τ_y can be written as a sum of stresses:

$$\tau_x=\tau_{zx}+\tau_{wu} \qquad \tau_y=\tau_{zy}+\tau_{wv} \qquad (3)$$

where τ_{zx} and τ_{zy} are viscosity shear stresses and τ_{wu} and τ_{wv} are turbulent shear stresses (Reynolds stresses). For atmospheric movements we can neglect the viscosity stresses in respect with Reynolds stresses, and the relations become:

$$\tau_x\approx\tau_{wu}=-\rho\overline{wu} \quad \text{and} \quad \tau_y\approx\tau_{wv}=-\rho\overline{wv} \qquad (4)$$

where u , v and w are the pulsation velocity components following the x , y and respectively z axis.

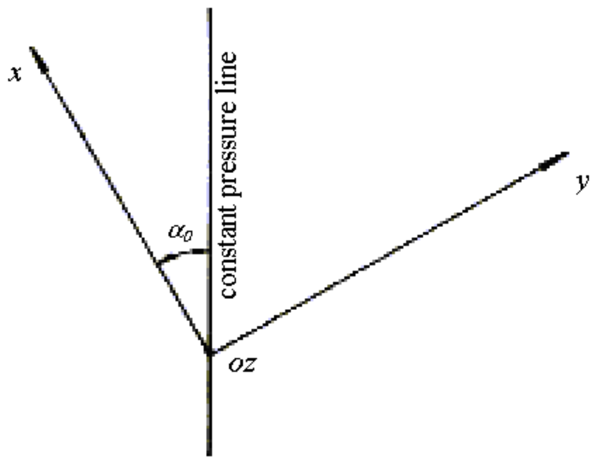

Fig. 1. The coordinate system chosen for mathematical modeling or air movement in the atmospheric boundary layer

If we differentiate the third movement equation from the equation system (1) in respect with x or y , these relations will result:

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right)+g\frac{\partial \rho}{\partial x}=0 \quad \text{and} \quad \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial y}\right)+g\frac{\partial \rho}{\partial y}=0 \tag{5}$$

which lead to the conclusion that the vertical variation (on z) of horizontal gradient pressure depends on the horizontal density gradient.

Making the acceptable hypothesis that the horizontal density gradient is negligible, it results:

$$\frac{\partial p}{\partial x}=0 \quad \text{and} \quad \frac{\partial p}{\partial y}=0$$

which is valid for barotropic currents with $\rho=\rho(p)$; the following expression results:

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right)=0 \quad \text{and} \quad \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial y}\right)=0 \tag{6}$$

from this results that $\partial p/\partial x$ and $\partial p/\partial y$ don't vary with the height z. From those shown above, we can conclude that the horizontal pressure gradient $\partial p/\partial n$ (where n is the normal in the horizontal plan to isobars) doesn't vary with the height z and we have the parameter:

$$\frac{\partial p}{\partial n}=\rho\left(fV_{gr}\pm\frac{V_{gr}^2}{r}\right) \tag{7}$$

where V_{gr} is the geo cyclostrophic wind velocity, and r is the radius of the isobars. If we assume the simplified case of linear and parallel isobars, the wind outside the ABL is the geostrophic wind, and the velocity measure is $G=(\partial p/\partial n)/\rho f$. Noting with U_g and V_g the $\overline{G}$ velocity components in respect with x and respectively y axis, results:

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=fV_g \quad \text{and} \quad \frac{1}{\rho}\frac{\partial p}{\partial y}=-fU_g \tag{8}$$

where $\partial p/\partial x$ and $\partial p/\partial y$ are the components on z and respectively y axis, of the horizontal pressure gradient $\partial p/\partial n$.

Substituting the expressions (5-8) in the two equations of the system (5-1), it results the equations for atmospheric boundary layer for geostrophic wind in the hypothesis of barotropic currents:

$$\begin{aligned}
U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}&=-f\left(V_g-V\right)+\frac{1}{\rho}\frac{\partial \tau_x}{\partial z}\\
U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}&=f\left(U_g-U\right)+\frac{1}{\rho}\frac{\partial \tau_y}{\partial z}
\end{aligned} \tag{9}$$

The limit conditions used for integrating the equation system are:

- at earth level (surface) ($z = 0$) the wind velocity is zero;
- at the upper limit of atmospheric boundary layer ($z = \delta$), the wind velocity is G, and the shear stress is zero.

3. CLOSING THE EQUATION SYSTEM OF ATMOSPHERIC BOUNDARY LAYER

In the following paragraphs we present a few models of turbulent flow which will be able to emphasise the problem of closure. They consist in introducing a formal identity (turbulent viscosity, mixt length etc.), easier to imagine than the random field of fluctuations in the real flow.

3.1. Closing the field of average velocity

In the average movement equation there are a series of supplementary terms (τ_x, τ_y) which are supplementary unknown terms in the equation. To solve these equations we introduce calculus expression for these terms, as a function of average movement characteristics.

The most known model is based upon the turbulent viscosity coefficient hypothesis of Boussinesq:

$$\tau_x \approx \tau_{wu} = -\rho\overline{wu} = \rho v_t \frac{\partial U}{\partial z} \tag{10}$$

$$\tau_y \approx \tau_{wv} = -\rho\overline{wv} = \rho v_t \frac{\partial V}{\partial z} \tag{11}$$

where v_t is the turbulent viscosity coefficient.

Another model is based upon the mixture length theory of Prandtl, which introduces the notion of mixture length and which takes into account the hypothesis of conserving movement quantities along the movement. The calculus relations for turbulent shear stresses are:

$$\tau_x = -\rho l_m^2 \frac{\partial U}{\partial z}\sqrt{\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2} \tag{12}$$

$$\tau_y = -\rho l_m^2 \frac{\partial V}{\partial z}\sqrt{\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2} \tag{13}$$

where l_m is Prandtl the mixture length.

3.2. Closing an average turbulent field

To realise this type of closure, we are using the kinetic energy conservation (energy balance) for turbulent velocity fluctuations, which can be written as follows:

$$U\frac{\partial}{\partial x}\left(\frac{1}{2}\overline{q^2}\right)+V\frac{\partial}{\partial y}\left(\frac{1}{2}\overline{q^2}\right)+W\frac{\partial}{\partial z}\left(\frac{1}{2}\overline{q^2}\right)-\left(\frac{\tau_x}{\rho}\frac{\partial U}{\partial z}+\frac{\tau_y}{\rho}\frac{\partial U}{\partial z}\right)+ \\ +\frac{\partial}{\partial z}\overline{\left[w\left(\frac{p'}{\rho}+\frac{1}{2}q^2\right)\right]}+\varepsilon=0 \tag{14}$$

The lines that appear above different terms from the above equation represent their averaging over a period of time. The resultant fluctuant velocity is noted with q and is expressed by the relation:

$$q=\sqrt{u^2+v^2+w^2} \tag{15}$$

where u, v and w represent fluctuating velocity components along x, y and respectively z axis, and p' is the fluctuating pressure

The terms in the turbulent kinetic energy balance equation have the following signification: (1) – convection term, (2) - turbulent energy production term, (3) – diffusion term, (4) – dissipation term.

Using the turbulent kinetic energy balance equation, along with other phenomenological relations, attached to Reynolds equations and with the continuity equation, represents the closing of turbulent average field.

The closing of the average turbulent field was used with very good results in the study of the tri-dimensional boundary layer. In order to do so, the supplementary relations were proposed:

$$\sqrt{\tau_x^2+\tau_y^2}=\rho a_1\overline{q^2}\,;\quad \overline{w\left(\frac{p'}{p}+\frac{1}{2}q^2\right)}=\frac{1}{V_{gr}}\left(\overline{q^2}\right)_{\max}\overline{q^2}a_2\frac{y}{\delta} \\ \varepsilon=\frac{\sqrt{\left(\overline{q^2}\right)^3}}{l_d\frac{y}{\delta}}\,;\quad \frac{\tau_x}{\frac{\partial U}{\partial z}}=\frac{\tau_y}{\frac{\partial V}{\partial z}} \tag{16}$$

where δ is boundary layer thickness, V_{gr} velocity at the upper limit of boundary layer, $a_1=0.16$, a_2 diffusion function, and l_d dissipation length.

To accomplish the closing, the diffusion function a_2 and the dissipation length l_d must be specified. In order to do so, J. F. Nash proposes the fluctuations in respect with y/δ as in Figure 2 and available for three-dimensional turbulent boundary layer.

4. ATMOSPHERIC BOUNDARY LAYER STRUCTURE BASED ON MONIN-OBUKHOV THEORY

In the close proximity of the earth's surface, the boundary layer state is quasi-stationary and the effect of the deviating Coriolis force and of the pressure horizontal gradient is negligible. At heights z that can be considered large in respect with the roughness height $z_0(z>50z_0)$, the turbulent condition of the surface layer with the thickness of $0.1\,\delta$, is decided by the friction velocity U_*, the kinematical heat flow Q_0, the floatability parameter $\beta(\beta = g/T)$ and the roughness z_0.

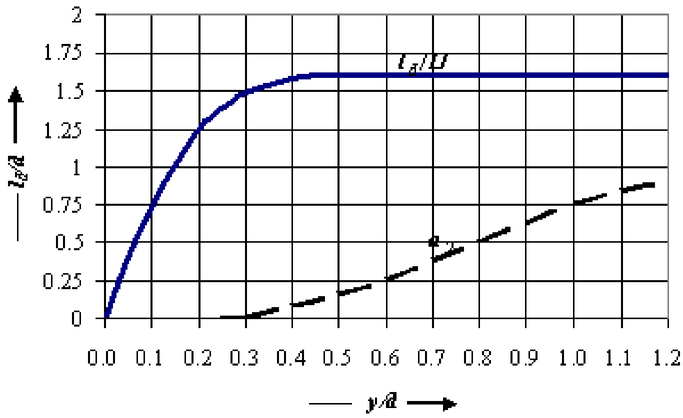

Fig. 2. Diffusion function a_2 variation and dissipation length l_d variation in respect with y/δ, for atmospheric boundary layer.

In the surface layer with a turbulent but constant flow, the wind direction is constant, and the length scale is the Monin-Obukhov length, given by the mathematical expression:

$$L = -\frac{U_*^3}{k\beta Q_0} \tag{17}$$

where k is Kármán constant ($k \cong 0{,}4$).

For $z \le 0{,}1|L|$, the turbulence is only governed by dynamic factors, and the average vertical profile velocity is expressed by the following logarithmic law:

$$U(z) = \frac{U_*}{k}\ln\frac{z}{z_0} \tag{18}$$

The mixture length l_m and the turbulent diffusion coefficient K_U vary linearly in respect to the height z, therefore: $l_m \sim k\,z$ and $K_U \sim k\,U_*\,z$.

For $z > 0{,}1|L|$, the turbulence is governed both by dynamic factors and thermal factors. The Monin-Obukhov similarity theory conducts to the following vertical profile of average velocity:

$$U(z_2) - U(z_1) = \frac{U_*}{k}\left[f_U\left(\frac{z_2}{L}\right) - f_U\left(\frac{z_1}{L}\right)\right] \tag{19}$$

where f_U and f_T are universal functions, for which, at a first approximation, the following vertical variation laws can be adopted:

- in a stable condition ($L > 0$), a linear logarithmic law:

$$f_U \cong f_T \cong \ln\frac{z}{L} + 10\frac{z}{L} \qquad (20)$$

- in an unstable condition ($L < 0$), a free-convection law:

$$f_U \cong f_T \cong 1.2\left(\frac{z}{L}\right)^{-1/3} + 0.25 \qquad (21)$$

In the layer situated above the surface layer, named transitory layer, the hypothesis of quasi-stationary is not generally valid, and the similarity theory doesn't give satisfactory results.

The forces given by horizontal pressure gradient and the deviating Coriolis force due to earth's rotation, influence both the wind velocity and the turbulent state (behaviour), thus the average wind velocity suffers one rotation when the altitude z increases, in regard to the surface shear stress τ_0, to the right in the north hemisphere, and to the left in the south hemisphere. In the area of the atmospheric boundary layer, the velocity hodograph (vertical velocity curve) has the shape of a logarithmic spiral.

5. CONCLUSIONS

The atmospheric boundary layer is a formation at meteorological scale wherein the air movement is mostly turbulent. In the outer area of atmospheric boundary layer the turbulence at reduced scale doesn't manifest except in an intermittently manner.

The model presented in this paper belongs to the theoretical study of the atmospheric boundary layer and is in accordance with the actual theoretical and experimental researches. The study is limited to the horizontal and homogenous case, named planetary boundary layer, which is equivalent to the homogenous and horizontal atmospheric boundary layer. Virtually, this restriction is justified only in the case wherein the statistic characteristics vary less than 10% along horizontal distances of size of 10δ.

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