# NUMERICAL SIMULATION OF ELASTIC TORSIONAL CONTACT WITH SLIP AND STICK

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**Abstract:** The paper presents an algorithm based on the conjugate gradient method, employed to simulate the slip-stick elastic contact under normal and tangential loading, when the latter consists of a moment of torsion smaller than the one inducing gross-slip. While the algorithm can be applied to any contact geometry with small slope (as to assimilate the contacting bodies with elastic half-spaces and to apply Boussinesq and Cerruti fundamental solutions), program validation is performed in the case of a spherical contact undergoing torsion applied simultaneously to a normal force. Numerical predictions match well existing closed-form solution for the corresponding contact scenario.

## 1. INTRODUCTION

The concentrated mechanical contact often involves a combination of normal contact and a relative motion between contacting surfaces, such as sliding, spinning or rolling. The situation when normal force, largely investigated due to existence of Hertz contact solution, is coupled with torsional loading, was covered by the works of Lubkin, [8], Hetenyi and McDonald, [4] and Hills and Sackfield, [5]. Based on the results of Mindlin, [9], Johnson, [6], presents the solution of a sphere in torsion, when a partial slip regime is established on the contact area. This solution is later reviewed and enhanced for the case of viscoelastic materials by Dintwa et al., [1]. Gallego and Nélias, [2], and Gallego, Nélias, and Deyber, [3]. advance a numerical model for the elastic contact under various loading types, and apply it to fretting modes I, II and III.

A numerical algorithm based on the conjugate gradient method for the slip-stick torsional contact is advanced in this paper. Its predictions are matched with existing analytical solution for the case of an elastic sphere in torsion.

## 2. OVERVIEW OF EXISTING ANALYTICAL SOLUTION

A cylindrical coordinate system $(r,\theta,x_3)$ is considered. A torsional moment $M_3$ is applied in addition to a normal force $W$, both acting along direction of $\vec{x}_3$, in a contact with Hertz contact geometry. Contact radius $a$, and pressure distribution, $p(r)$, are assumed to be known from Hertz theory. If the torsional moment $M_3$ is smaller than a limiting value which induces gross-slip, an annulus of slip penetrates from contact area boundary, enveloping a stick area of radius $a_S$. The shear circumferential stress $q_\theta(r)$ arising due to application of $M_3$, yields from the following equation:

$$q_\theta(r)=\begin{cases}\dfrac{3\mu W}{2\pi a^2}\sqrt{1-(r/a)^2}\left[\dfrac{\pi}{2}+k^2D(k)F(k^*,\varphi)-K(k)E(k^*,\varphi)\right], & r\le a_S;\\ \dfrac{3\mu W}{2\pi a^2}\sqrt{1-(r/a)^2}, & a_S\le r\le a,\end{cases} \tag{1}$$

where:

$$k^* = a_S/a\,;\; k = \sqrt{1-(k^*)^2}\,;\; \varphi = \arcsin\left[\frac{1}{k^*}\sqrt{\frac{(k^*)^2-(r/a)^2}{1-(r/a)^2}}\right]. \tag{2}$$

$F(k^*,\varphi)$ and $E(k^*,\varphi)$ are the incomplete elliptical integrals of first and of the second kind, of modulus $k^*$ and amplitude $\varphi$, $K(k)$ and $E(k)$ the complete elliptical integrals of the first and of the second kind, and

$$D(k) = \left(K(k)-E(k)\right)/k^2\,. \tag{3}$$

The rigid-body angle of torsion $\phi_3$ and the moment of torsion $M_3$ yield from the following equations:

$$\phi_3 = \frac{3\mu W}{4\pi G a^2}k^2 D(k)\,; \tag{4}$$

$$M_3 = \frac{\mu W a}{4\pi}\left\{\frac{3\pi^2}{4} + k^* k^2\left[6K(k) + \left(4(k^*)^2-3\right)D(k)\right] - 3kK(k)\arcsin(k^*)\ldots\right.$$
$$\left. -3k^2\left[K(k)\int_0^{\pi/2}\frac{\arcsin(k^*\sin\alpha)d\alpha}{(1-(k^*)^2\sin^2\alpha)^{3/2}} - D(k)\int_0^{\pi/2}\frac{\arcsin(k^*\sin\alpha)d\alpha}{(1-(k^*)^2\sin^2\alpha)^{1/2}}\right].\right. \tag{5}$$

with G the shear modulus. The limiting value for the torsional moment, namely the moment for which gross-slip is imminent, can be expressed as:

$$M_{3\,\lim} = \frac{3\pi}{16}\mu W a. \tag{6}$$

The following relation holds, [1]:

$$\frac{G a^2 \phi_3}{\mu W} = \frac{1}{8}\left[1-\left(1-\frac{3}{2}\frac{M_3}{\mu W a}\right)^{1/2}\right]\left[3-\left(1-\frac{3}{2}\frac{M_3}{\mu W a}\right)^{1/2}\right]. \tag{7}$$

## 3. MODEL FORMULATION

The principles of contact problem digitization are detailed elsewhere, [10]. The nodal value of every continuous distribution is denoted by $f(i,j)$, where $i$ and $j$ are two indices used to identify the position of the rectangular patch within the imposed mesh. Also, at this point, solution of normal contact problem is assumed to be known from Hertz theory, or can be obtained for any contact geometry using the algorithm advanced by Polonsky and Keer, [10].

If a slip-stick regime is assumed to occur when applying the torsional moment, any cell in the contact area $A_C$ will be in either slip or stick regime. On the stick area $A_S$, a static friction regime is established, with vanishing relative slip distances $\mathbf{s}(s_1,s_2)$ between corresponding points on the limiting surfaces, as opposed to the slip area, where friction is kinetic, and the norm of shear traction $\|\mathbf{q}(q_1,q_2)\|$ equals pressure multiplied by the friction coefficient $\mu$, assumed constant over all contact area.

The discrete model for the elastic contact problem with torsional moment under partial slip regime consists of the following relations:

$$\begin{bmatrix} s_1(i,j) \\ s_2(i,j) \end{bmatrix} = \begin{bmatrix} u_1(i,j) \\ u_2(i,j) \end{bmatrix} - \phi_3 \begin{bmatrix} x_2(i) \\ x_1(j) \end{bmatrix}, (i,j) \in A_C ; \tag{8}$$

$$\sum_{(i,j)\in A_C} \left[ x_1(j) q_2(i,j) + x_2(i) q_1(i,j) \right] = M_3 / \Delta ; \tag{9}$$

$$\|\mathbf{q}(i,j)\| \le \mu p(i,j), \|\mathbf{s}(i,j)\| = 0 , (i,j) \in A_S . \tag{10}$$

$$\|\mathbf{q}(i,j)\| = \mu p(i,j), \|\mathbf{s}(i,j)\| > 0 ,\ (i,j) \in A_C - A_S . \tag{11}$$

where $\Delta$ is the elementary cell area, and $\phi_3$ the rigid-body angle of torsion.

Eq. (8) is the geometric condition of deformation, relation (9) establish the static force equilibrium on the tangential direction, and Eqs. (10) and (11) are the complementarity conditions, assessing the slip or stick regime of every cell in the contact area.

As contacting bodies are assimilated with elastic half-spaces, elastic displacements entering Eq. (8) can be computed by applying superposition principle to Boussinesq and Cerruti fundamental solutions. All three contact stresses, namely pressure $p$ and shear tractions $q_1$ and $q_2$ contribute to displacements $u_1$ and $u_2$:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \end{bmatrix} \otimes \begin{bmatrix} q_1 & q_2 & p \end{bmatrix}^T . \tag{12}$$

where symbol "$\otimes$" denotes discrete cyclic convolution. The Discrete Convolution Fast Fourier Transform technique, [7], can be used to speed up the computation. The influence coefficients $K_{ij}(k,\ell)$, expressing displacement in the cell $(k,\ell)$, along the direction of $\vec{x}_i$, due to a unit traction acting in origin along $\vec{x}_j$, result from integration over a rectangular domain of fundamental solutions for the elastic half-space.

The unknowns of the newly formulated problem are the stick area $A_S$ and the distribution of shear tractions $\mathbf{q}$. As system (8) - (11) is similar to the one describing the elastic contact problem undergoing normal load, the same conjugate gradient approach should be used. Indeed, Eq. (8) can be considered as a linear system which must be solved on the stick area, having the nodal values of the shear tractions as unknowns. As system matrix, namely the influence coefficients matrix from Eq. (12), is symmetric and positive definite, conjugate gradient method is guaranteed to converge to system solution. The size of the system to be solved, which is also unknown, as stick area is a priori unknown, is also adjusted during iterations, according to complementarity conditions (10) and (11).

Problem solution results as the solution of the linear system resulting from digitization of condition of deformation (Eq. (8)). This solution is also adjusted, at every conjugate gradient iteration, according to static force equilibrium, Eq. (9). The conjugate directions are reinitialized every time the size of the system is modified due to enforcement of complementarity conditions.

## 4. ALGORITHM OVERVIEW

The numerical approach proposed herein is based on the notorious algorithm advanced by Polonsky and Keer, [10], for the elastic rough contact undergoing normal loading. The initial guess for shear tractions is adopted in the form:

$$\begin{cases} q_1(i,j) = x_2(i)\cdot a; \\ q_2(i,j) = x_1(j)\cdot a. \end{cases} \tag{13}$$

where the unknown term $a$ is computed from static force equilibrium, Eq. (9), assuming all cells in the contact area are initially in stick regime. Auxiliary variables, and descent direction, are initialized: $\theta = 0$, $S_0 = 1$, $\mathbf{d}(i,j) = \mathbf{0}, (i,j) \in A_C$. The following operations are looped until convergence is reached:

1. Compute displacement field over domain of analysis $D$, using Eq. (12). Contribution of contact pressure does not change during iterations; therefore, it should be only computed once before entering the loop.

2. Estimate the angle of torsion $\phi_3$. For every cell in the stick area, the following equality holds:

$$\begin{bmatrix} u_1(i,j) \\ u_2(i,j) \end{bmatrix} - \phi_3 \begin{bmatrix} x_2(i) \\ x_1(j) \end{bmatrix} = 0, (i,j) \in A_S; \tag{14}$$

If the following functional is defined:

$$\aleph(\phi_3) = \left[u_1(i,j) - \phi_3 \cdot x_2(i)\right]^2 + \left[u_2(i,j) - \phi_3 \cdot x_1(j)\right]^2, (i,j) \in A_S, \tag{15}$$

then the torsional angle that verify best system (8) is found by setting the partial derivative of $\aleph$ to zero, leading to:

$$\phi_3 = \frac{\sum_{(i,j)\in A_S} \left[u_1(i,j)x_2(i) + u_2(i,j)x_1(j)\right]}{\left(x(j)\right)^2 + \left(y(i)\right)^2}. \tag{16}$$

3. Start the conjugate gradient loop. To this end, compute the relative slip distances using Eq. (8), and then the square sum of **s** components on the stick area:

$$S = \sum_{(i,j)\in A_S} \left[s_1^2(i,j) + s_2^2(i,j)\right]. \tag{17}$$

4. Compute the descent direction **d** in the conjugate gradient algorithm:

$$\begin{bmatrix} d_1(i,j) \\ d_2(i,j) \end{bmatrix} = \begin{cases} \begin{bmatrix} s_1(i,j) \\ s_2(i,j) \end{bmatrix} + \dfrac{S\theta}{S_0}\begin{bmatrix} d_1(i,j) \\ d_2(i,j) \end{bmatrix}, (i,j) \in A_S; \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^T, (i,j) \in A_C - A_S. \end{cases} \tag{18}$$

5. Memorize the value of $S$ for subsequent computation of descent direction: $S_0 = S$.

6. Assess the length of the step to be made along the descent direction:

$$\alpha = \frac{\begin{bmatrix} s_1 & s_2 \end{bmatrix} \cdot \begin{bmatrix} d_1 & d_2 \end{bmatrix}^T}{\begin{bmatrix} d_1 & d_2 \end{bmatrix} \cdot \left[ \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \otimes \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right]} . \tag{19}$$

7. Memorize the current solution to compute the relative error between consecutive iterations:

$$\mathbf{q}_0 = \mathbf{q} . \tag{20}$$

8. Update the system solution by making a step of length $\alpha$ along the direction of $\mathbf{d}$:

$$\mathbf{q} \leftarrow \mathbf{q} - \alpha \mathbf{d} . \tag{21}$$

9. Verify complementarity conditions in Eqs. (10) and (11) to determine the stick or slip status of every cell in the contact area. Cells for which Coulomb friction law is not verified are removed from the stick region, and the corresponding shear tractions are set to the value of static friction. In the same time, cells having micro-slip vectors $\mathbf{s}^{(ij)}$ not opposite to corresponding shear tractions are removed from the slip region and included in $A_S$:

$$\mathbf{q}(i,j) \leftarrow \mu p(i,j) \frac{\mathbf{q}(i,j)}{\left\| \mathbf{q}(i,j) \right\|} ,\ (i,j) \in \left\{ (i,j) : \left\| \mathbf{q}(i,j) \right\| > \mu p(i,j) \right\} , \tag{22}$$

$$A_S \leftarrow A_S - \left\{ (i,j) : \left\| \mathbf{q}(i,j) \right\| > \mu p(i,j) \right\} \cup \left\{ (i,j) : \mathbf{q}(i,j) \mathbf{s}(i,j) > 0 \right\} . \tag{23}$$

10. Impose static equilibrium by adjusting the current solution:

$$\begin{bmatrix} q_1(i,j) \\ q_2(i,j) \end{bmatrix} \leftarrow \begin{bmatrix} q_1(i,j) \\ q_2(i,j) \end{bmatrix} + \begin{bmatrix} x_2(i) \cdot b \\ x_1(j) \cdot b \end{bmatrix} ,\ (i,j) \in A_S , \tag{24}$$

where the unknown coefficient $b$ is computed by plugging Eq. (24) into Eq. (9):

$$b = \frac{M_3 / \Delta - \sum_{(i,j) \in A_C} \left[ q_2(i,j) x_1(j) + q_1(i,j) x_2(i) \right]}{\left( x_1(j) \right)^2 + \left( x_2(i) \right)^2} . \tag{25}$$

11. Verify convergence criterion:

$$\sum_{(i,j) \in A_C} \left\| \mathbf{q}^{(ij)} - \mathbf{q}_0^{(ij)} \right\| \le \varepsilon . \tag{26}$$

## 5. PROGRAM VALIDATION

To check the predictions of the newly advanced computer program, a spherical indenter of radius $R = 18mm$ is statically pressed with a normal force $W = 1kN$ against an elastic half-space, having the following elastic parameters: Young modulus $E = 210GPa$, Poisson's ratio $\nu = 0.3$. An increasing torsional moment $M_3 \le M_{3\,\lim}$ is applied. Dimensionless coordinates are defined as ratio to Hertz contact radius, $\bar{x}_i = x_i / a$, $i = 1,2$, dimensionless moment of torsion $\bar{M}_3 = M_3 / M_{3\,\lim}$, and dimensionless shear tractions as ratio to hertzian pressure, $\bar{q}_i = q_i / p_H$.

Two-dimensional distributions of shear tractions $\bar{q}_1$ and $\bar{q}_2$, when $\bar{M}_3 = 0.9$, are presented in Fig. 1. One can see that the distribution of one parameter can be obtained by applying to the other a 90 degree rotation with respect to the central axis of the contact. Two-dimensional distribution of dimensionless norm of shear tractions $\|\mathbf{q}(i,j)\| = \sqrt{q_1^2(i,j) + q_2^2(i,j)} / p_H$ is presented in Fig. 2. Radial profiles of this distribution for two loading levels are compared in Fig. 3 with the analytical counterparts given by Eq. (1), and a good agreement is found.

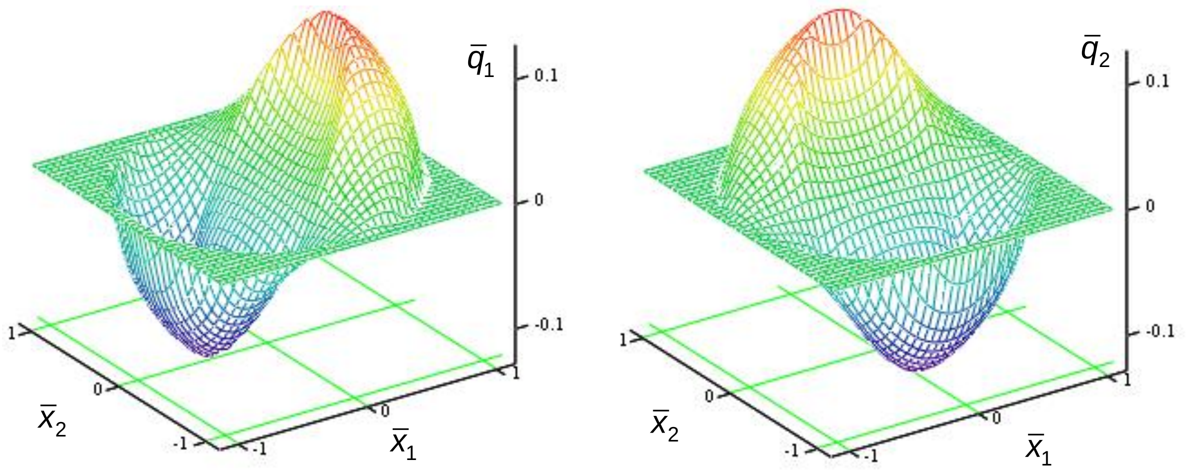


***Fig. 1. Dimensionless shear tractions*** $\bar{q}_1$ ***and*** $\bar{q}_2$***,*** $\bar{M}_3 = 0.9$

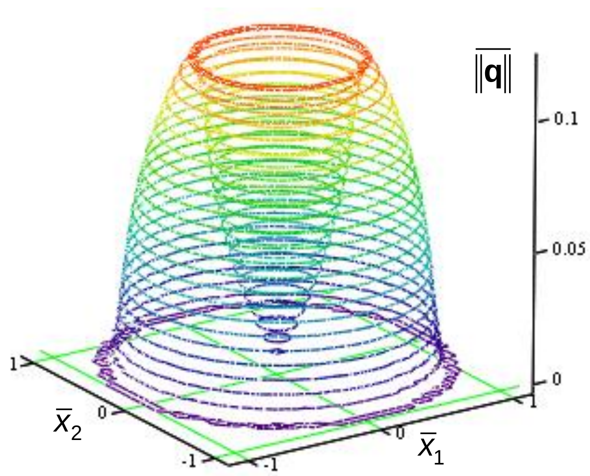


***Fig. 2. Dimensionless norm of shear tractions,*** $\bar{M}_3 = 0.9$

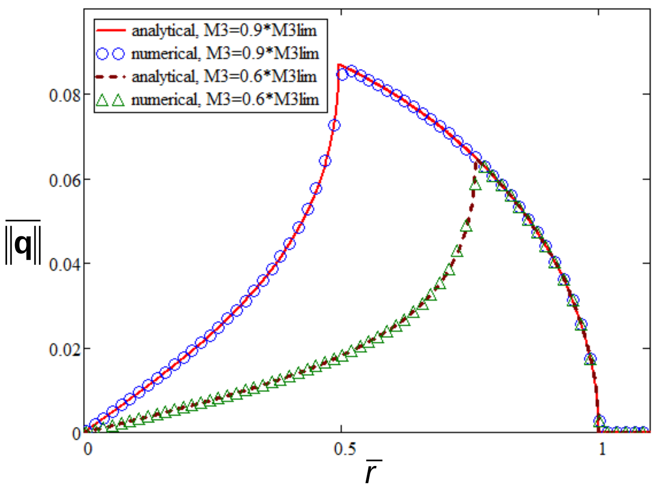


***Fig. 3. Profiles of dimensionless norm of shear tractions***

Numerical predictions for the stick radius are compared in Fig. 4 with the results obtained by solving Eq. (5) in $k^*$ as a transcendent equation, using appropriate numerical methods. In Fig. 5, the points obtained numerically overlap the analytical curve described by Eq. (7), although differences of up to a few percents can be seen, which can be partially attributed to discretization error or to a poor choice of imposed precision in convergence criterion (26).

The overall agreement between program predictions and analytical results is considered as satisfactory given the complexity of the problem to be solved.

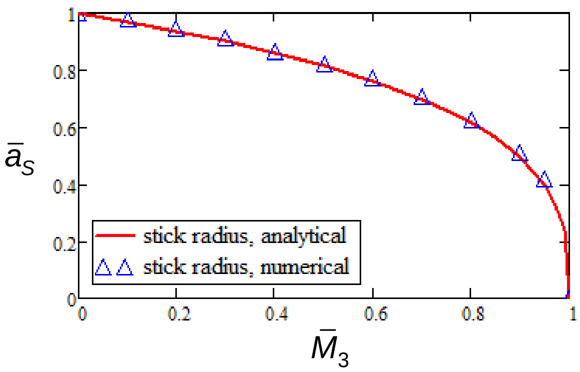


***Fig. 4. Dimensionless stick radius versus loading level***

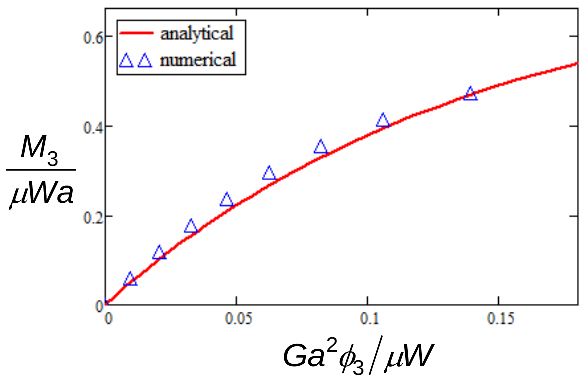


***Fig. 5. Variation of angle of torsion***

## 6. CONCLUSIONS

A numerical program to simulate the elastic torsional contact with slip and stick is presented in this paper. Numerical predictions are matched against closed-form solution in the case of a sphere in torsion and a good agreement is found.

Shear tractions distribution yields as the solution of the linear system arising from digitization of integral condition of deformation. This system is solved numerically using the conjugate gradient method. The size of this system depends upon the stick area, which is iterated according to complementarity conditions. An additional constriction is enforced by imposing the static force equilibrium.

As the proposed algorithm is based on one level of iterations only, the computational efficiency is high, mainly due to rapid decrease of residual in the conjugate gradient method.

Numerical predictions of circumferential shear tractions, of stick radius and of rigid-body angle of torsion, match well existing closed-form expressions for the sphere in torsion.

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