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STABILITY IN A CLASSIC MODEL FOR ECONOMIC GROWTH

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Abstract: The paper emphasizes some stability issues for the solutions of the evolution equation that describes the dynamics in the Solow-Swan classic growth model (see [3] and [5]), as an extension of the studies developed in [4]. The mathematical framework offers the instruments for the study of stability, which enables the decision maker to complete the choice of economic strategies.

1. INTRODUCTION

The economic growth, as a complex evolution process on long term is characterized by the increasing of its typical dimensions and by the transformation of the society structures. The economic growth is the final aim of all strategies. This priority responds to a double concern of the nations: on one hand, to withstand the constant development of individual and collective needs, concretized in the growth of the national welfare and, on the other hand, in the confrontation on international level. The economic growth is thus defined as the move and the profound changes of an economy in its whole.

In order to choose an investment policy, a stability study is appropriate. The mathematic framework, where the definition of stability and some characterizations are emphasized (see also [1] and [2]), offers the tools to examine two solutions of the evolution equation that arises in the economic model, to decide which the equilibrium points are, and which economic strategies are interesting to be followed.

2. MATHEMATIC FRAMEWORK

Let us consider the system of differential equations of order 1

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, ..., x_n),\ i = \overline{1,n} \tag{2.1}$$

where $t_0 \geq 0$ is given, $f_i : [t_0, \infty) \times D \to \mathbf{R}, D \subset \mathbf{R}^n$, and D is a domain. The variable $t \geq t_0$ denotes the time variable and the vector $x = (x_1, x_2, ..., x_n)$ denotes the coordinates of a point M in D. We consider that the functions f_i, $i = \overline{1,n}$, are continuous with continuous partial derivatives of order 1 on $[t_0, \infty) \times D$, relative to each variable. Hence they satisfy the standard conditions for the existence and uniqueness of the solutions of system (2.1).

Let us denote $x_i(t_0) = x_i^0$, $i = \overline{1,n}$, the initial conditions and $x_i = x_i(t; t_0, x_1^0, x_2^0, ..., x_n^0)$, $i = \overline{1,n}$ the coordinates of the vector $x = (x_1, x_2, ..., x_n)$ relative to the initial conditions.

Definition 2.1. The solution of the system (2.1) is said to be *stable* if for each $\varepsilon > 0$ there exists a positive number $\delta(\varepsilon)$ such that following relation is true

$$\left| x_i(t; t_0, x_1^0, ..., x_n^0) - x_i(t; t_0, y_1^0, ..., y_n^0) \right| < \varepsilon, \tag{2.2}$$

for all $t \geq t_0$, where $\left| y_i^0 - x_i^0 \right| < \delta(\varepsilon)$, for all $i = \overline{1,n}$.

Definition 2.2. The solution of the system (2.1) is said to be *instable* if it is not stable, i.e. there exists a value $\varepsilon_0 > 0$, a moment $t = T$ and an $i_0 \in \{1,2,..., n\}$ such that, for all positive numbers δ, following relation holds

$$\left|x_{i_0}(t;t_0,x_1^0,...,x_n^0) - x_{i_0}(t;t_0,y_1^0,...,y_n^0)\right| \geq \varepsilon_0$$

for all $t \geq t_0$, where $\left|y_i^0 - x_i^0\right| < \delta(\varepsilon)$, for all $i = \overline{1,n}$.

Remark 2.1. The notion of stability was defined in Definition 2.1 at the moment $t = t_0$. On the other hand, in the case of uniform stability, constant δ does not depend on t_0, and, hence, relation (2.2) holds for all $t_0 \geq 0$.

According to the direct Liapunov method, the stability of the solution $x = (0,0,...,0)$ of the system (2.1), can be studied without solving the differential equations. Following theorem presents sufficient stability conditions.

Theorem 2.1. *Let us consider $t_0 \geq 0$. If there exists a mapping $v: D \subset \mathbf{R}^n \to \mathbf{R}$, continuously differentiable, that satisfies in a neighborhood of the origin following conditions:*

(i) function v has a strict minimum in the origin, i.e. $v(x) \geq 0$, $\forall x = (x_1,x_2,\ldots,x_n) \in D$ and $v(x) = 0$ if and only if $x_i = 0$, $\forall i = \overline{1,n}$;

(ii) the derivative of function v, along the integral curves of the system (2.1), *satisfies the relation*

$$v'(x) = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(t,x_1,x_2,...,x_n) \leq 0, \ \forall t \geq t_0, \ \forall x = (x_1,x_2,...,x_n) \in D. \qquad (2.3)$$

Then, the solution $x = (0,0,...,0)$ of the system (2.1) *is stable.*

Proof. Let us consider $\varepsilon > 0$ and $\rho \in (0,\varepsilon]$ such that $B_\rho = \{x \in \mathbf{R}^n, \|x\| \leq \rho\} \subset D$. Let us denote $\alpha = \min_{\|x\|=\rho} v(x)$ and let $\beta \in (0,\alpha)$. We will define the set $\Omega_\beta = \{x \in B_\rho, v(x) \leq \beta\}$. As $v'(x(t)) < 0$, for all $t \geq t_0$, the inequality $v(x(t)) \leq v(x(t_0)) \leq \beta$ holds for all $t \geq t_0$. Hence $x(t_0) \in \Omega_\beta$ implies that $x(t) \in \Omega_\beta$ for all $t \geq t_0$, There exists $\delta > 0$ such that $\|x\| < \delta$, which implies $v(x) \leq \beta$. It follows that $B_\delta \subset \Omega_\beta \subset B_\rho$. We have $x(t_0) \in B_\delta$, which implies $x(t_0) \in \Omega_\beta$. It follows that $x(t) \in \Omega_\beta$ and $x(t) \in B_\rho$, for all $t \geq t_0$. We obtain that $\|x\| < \varepsilon$ and $\|x(t)\| < \varepsilon$, for all $t \geq t_0$, as $\|x(t_0)\| < \delta$. Hence, the null solution of the system (2.1) is stable.

Remark 2.2. The mapping *v* is called a *Liapunov function*. Its derivative along the integral curves of the system (2.1) depends on the differential equations. If we denote the solution of the system $\varphi(t,x_1,x_2,...,x_n)$, then

$$v'(x) = \frac{d}{dt} v(\varphi(t,x_1,x_2,...,x_n)) \big|_{t=t_0}. \qquad (2.4)$$

Remark 2.3. Theorem 2.1 can be applied in the study of stability of other solutions of the system (2.1) than $x = (0,0,...,0)$ by means of an appropriate translation of the origin of the coordinates system, a specific one for every solution.

Remark 2.4. In the particular case of systems of homogenous linear differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x, \ i = \overline{1,n},$$

where $a_{ji}(t) = -a_{ij}(t), i \neq j$ and $a_{ii}(t) \leq 0,$ we can consider the Liapunov function defined by the relation $v(x) = \sum_{i=1}^{n} x_i^2$, where $x = (x_1,x_2,...,x_n)$.

Definition 2.3. Let us consider $t_0 \geq 0$. A vector $x_e \in D$ is an *equilibrium* for the system of differential equations (2.1) at moment t_0 if the relation $f(t, x_e) = 0$ holds for all $t \geq t_0$.

Remark 2.5. If $x_e \in D$ is an *equilibrium* for the system (2.1) at moment t_0, then it is and equilibrium point at any moment $t \geq t_0$.

Remark 2.6. Theorem 2.1 holds for $x_e = 0$.

3. ECONOMIC FRAMEWORK

Let us describe the economic relations of the model. The capital stock $K(t)$ and the labor $L(t)$ are inputs. The production function, which gives the gross national product, is given, at any moment t, by the relation $Y(t) = F(K(t), L(t)),\ \forall t \geq 0$. If we consider the per capita variables $y = YL^{-1}$ and $k = KL^{-1}$, the previous relation becomes $y(t) = f(k(t))$, $t \geq 0$.

The equilibrium on macroeconomic level is given by $I = S$, where I denotes the aggregate demand and S the aggregate supply.

The consumption function is given by the relation $C = cY$, where c is the marginal propensity for consumption and $c = 1-s$, s being the propensity for saving. The population growth rate is denoted by n.

The properties of the production function F are, as given in the classic model:

(i) function F is homogenous and linear, which indicates constant productivity

(ii) function F is positively defined and the lack of one of the factor implies a null value

$$F(K(t), 0) = 0 \text{ and } F(0, L(t)) = 0,\ \forall t \geq 0.$$

(iii) function f has positive partial derivatives of first order, representing the marginal productivity of capital, respectively of labor

$$F'_K(t) = \frac{\partial F(K,L)}{\partial K} > 0 \text{ and } F'_L(t) = \frac{\partial F(K,L)}{\partial L} > 0,\ \forall\ K, L > 0,\ \forall t \geq 0$$

(iv) function f has negative partial derivatives of second order

$$F''_{KK}(t) = \frac{\partial^2 F(K,L)}{\partial K^2} < 0,\ F''_{LL}(t) = \frac{\partial^2 F(K,L)}{\partial L^2} < 0,\ F''_{KL}(t) = \frac{\partial^2 F(K,L)}{\partial K \partial L} < 0,\ \forall\ K, L > 0, \forall t \geq 0$$

(v) $\lim_{K\to 0} F(K(t), L(t)) = \infty$ and $\lim_{K\to\infty} F(K(t), L(t)) = 0,\ \forall L > 0,\ \forall t \geq 0$.

Remark 3.1. From relations (i) and (iv) it is obvious that following statements hold:

(a) $\lim_{K\to 0} F'_L(K(t), L(t)) = 0$ and $\lim_{K\to\infty} F'_L(K(t), L(t)) = \infty,\ \forall L > 0,\ \forall t \geq 0$

(b) $\lim_{L\to 0} F'_K(K(t), L(t)) = 0$ and $\lim_{L\to\infty} F'_K(K(t), L(t)) = \infty,\ \forall K > 0,\ \forall t \geq 0$.

In what follows, the lowercases will indicate the characteristics defined per capita, where every element is divided by the level of labour.

Remark 3.2. The properties of the intensive production functions are derived from the properties (i) – (v):

(i)' $f'(k) = \frac{\partial F}{\partial K} > 0, \forall k > 0$;

(ii)' $f''(k) < 0, \forall k > 0$;

(iii)' $\lim_{k\to\infty} f(k) = 0$, $\lim_{k\to 0} f'(k) = \infty$.

In what follows, we will give the dynamics of the Solow-Swan model, obtained from the evolution equation that describes the dynamics of the capital in the Harrod-Domar model

$$K' = sY - dK,$$

where d is the depreciation rate of the capital K, considered constant.

We obtain following relation

$$\frac{k'}{k}+\frac{L'}{L}=s\frac{Y}{K}-d\,.$$

If we consider that the ratio population/labor force is constant, the growth relation for *k* is obtained

$$\frac{k'}{k}=s\frac{Y}{K}-d-n$$

equivalent with

$$k'=sy-(d+n)k\,. \qquad (3.1)$$

According to the previously stated properties, the fundamental relation of the economic dynamics in the Solow-Swan model is

$$k'=sf(k)-(d+n)k\,. \qquad (3.2)$$

4. THE MAIN RESULTS

In this section, two solutions for the evolution equation that describes the model are given, in the general case and in a particular case with a given function of production.

In the general case of a production function, the equilibrium condition $k'(t) = 0$ implies that there exists an equilibrium value k^* such that

$$sf(k^*)=(d+n)k^*\,. \qquad (4.1)$$

Proposition 4.1. *The differential equation* (3.2)*, which describes the dynamics in the Solow-Swan model, has a stable solution.*

Proof. Let us consider the equilibrium value k^* such that $sf(k^*) = (d+n)k^*$. We will define a function $v: \mathbf{R}_+ \rightarrow \mathbf{R}$ by $v(t) = [k(t) - k^*]^2$. Following relation holds

$$v'(t)= 2[k(t) - k^*][sf(k(t)) -(d+ n)\,k(t)].$$

According to Remark 3.2 (ii)', the intensive production function is concave. We obtain that *f'* is a nondecreasing function, and, hence,

$$f'(k^*)\geq\frac{f(k(t))-f(k^*)}{k(t)-k^*},\forall t\geq 0\,.$$

Further, following relation is obtained

$$v'(t)\leq-\frac{2(d+n)[k(t)-k^*]}{f(k^*)}[f(k^*) - k^*f'(k^*)]<0,$$

because $f(k^*) - k^*f'(k^*) > 0,$ as the marginal product of labor. According to Theorem 2.1, it follows that the ratio capital-labor, obtained from relation (4.1)

$$k^* = \frac{s}{d+n}f(k^*) > 0$$

is a stable solution for the evolution equation of the Solow-Swan growth model. □

Remark 4.1. For $k < k^*$, we obtain a growth of the capital as $k'(t) > 0$, and for $k > k^*$, we obtain a depreciation of the capital as $k'(t) < 0$.

In what follows, as a particular case, let us consider that the production function is given by the relation

$$Y = F(K,L) = K^{\alpha}L^{1-\alpha}\text{, where } \alpha \in[0, 1]$$

is the elasticity parameter. The intensive form of the production function is $y = f(k) = k^{\alpha}$.

We will also consider an exogenous index *P* for the technological progress that grows exponentially, the Cobb-Douglas type production function has following expression $Y = K^{\alpha}(LP)^{1-\alpha}$ where $L = L_0e^{ht}$, $L_0 > 0$ and $P = P_0e^{\gamma t}$, where $P_0 > 0,$ and h,γ,β are negative parameters. In the hypothesis we have considered s the propensity for saving with $s \in (0, 1)$. Let us denote $K(0) = K_0$.

The evolution equation of the Solow-Swan growth model becomes

$$k'(t) = sY - \beta K,$$

which can also be written as a Cauchy problem

$$\begin{cases} k'(t) = sk^{\alpha} - (n + \gamma + \beta)k \\ k_0 = \dfrac{K_0}{P_0 L_0} \end{cases} \tag{4.2}$$

The equilibrium condition $k'(t) = 0$ implies that there exists an equilibrium value k^* such that following relation holds

$$s(k^*)^{\alpha} = (n + \gamma + \beta)k^* . \tag{4.3}$$

Proposition 4.2. *Let us consider $n > 0$ and $\alpha \in (0, 1)$. The solution of the Cauchy problem* (4.2) *is stable and $\lim_{t \to \infty} k(t) = k^*$, where k^* is the equilibrium value of the capital-labor ratio.*

Proof. We will denote $\rho = (1 - \alpha)(n + \gamma + \beta)$. The differential equation that describes the dynamics in the growth model is of Bernoulli type. We consider the classic function transformation given by $u(t) = k(t)^{1-\alpha}$. We obtain a linear differential equation

$$u'(t) = -\rho u(t) + (1 - \alpha)s$$

with the solution given by

$$u(t) = \frac{k_0^{1-\alpha}}{e^{\beta t}} + \frac{s(1 - \alpha)}{\rho}(1 - e^{-\rho t})$$

which is a weighted average of the initial value and equilibrium value. Hence, the solution of the Solow-Swan equation is

$$k(t) = [u(t)]^{\frac{1}{1-\alpha}}$$

and, further, given by the relation

$$k(t) = \left[\frac{k_0^{1-\alpha}}{e^{\beta t}} + \frac{s}{n + \gamma + \beta}(1 - e^{-\rho t}) \right]^{\frac{1}{1-\alpha}}$$

where ρ represents the rate at which the economy converges to the equilibrium growth path. As in the proof of Proposition 4.1, the solution is stable.

If we take $n > 0$ and $\alpha \in (0, 1)$, then $\lim_{t \to \infty} k(t) = k^*$, where k^* is obtained from relation (4.3)

$$k^* = \left(\frac{s}{n + \gamma + \beta} \right)^{\frac{1}{1-\alpha}}$$

and represents the equilibrium value of the capital-labor ratio. □

Remark 4.2. At any moment $t > 0$, the expression for income the per capita is given by

$$y(t) = \left[\left(\frac{y_0}{P_0} \right)^{\frac{1-\alpha}{\alpha}} e^{-\rho t} + \frac{s}{n + \gamma + \beta}(1 - e^{-\rho t}) \right]^{\frac{\alpha}{1-\alpha}} .$$

As a conclusion, beginning from any capital-labor ratio $k > 0$, the solution of the evolution equation, that gives the dynamics in the considered growth model, will converge to the equilibrium value k^*.

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References

[1] Cristescu, G., Bota (Stoica), C., Ecuaţii diferenţiale şi cu derivate parţiale, Ed. Mirton, Timişoara, 2001.
[2] Megan, M., Proprietes qualitatives des systemes lineaires controlees dans les espaces de dimension infinie, Monografie Matematică, Nr. 32, Tip. Univ. din Timişoara, 1988.
[3] Solow, R.M., Growth Theory: An exposition, Oxford University Press, 1988.
[4] Stoica, C., A bisectorial model for economic growth, Annals of the University of Oradea, Fascicle of Management and Technologic Engineering, VII (XVII) (2008) 2778-2781.
[5] Swan, T.W. Economic growth and capital accumulation, Economic Record, Vol. 32 (2) (1956) 334-361.