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COMPUTER AIDED OPTIMIZATION TECHNIQUES BASED ON GENERALIZED CONVEX MAPS

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Abstract. Many times in multiple criteria optimization problems, the non-smoothness of the objective functions and of the admissible domain preserves some properties, which are enough to guarantee the existence of the solution. They are included in the rich class of of invexity properties, which bring useful properties in connection with approaching the vectorial optimization problems with generalized convex objective functions on generalized convex domain. The study was inspired by some multiple criteria optimization problem arising during modeling an ecologic-economic efficiency problem of a Romanian railway modernization project. A procedure of geometrical representation of the objective function by maps leads to a sequential approximation of the solution. A method of estimating the error is given.

Many times in multiple criteria optimization problems, the non-smoothness of the objective functions and of the admissible domain preserves some properties, which are enough to guarantee the existence of the solution. As an example, the classes of g-convex sets and g-convex functions are described in this paper. The g-convexity is a particular type of invexity (see [4], [7], [9]), which brings useful properties in connection with approaching the vectorial optimization problems with g-convex objective functions on g-convex domain. The study was inspired by some multiple criteria optimization problem arising during modeling an ecologic-economic efficiency problem of a modernization project in the Romanian railway network. A procedure of geometrical representation of the objective function by maps leads to a sequential approximation of the solution. A method of estimating the error is given.

First of all, let us remind that a set is convex whenever it contains the straight-line segment determined by each pair of its points. If (V, +, ·) is a real linear space then f: V $\rightarrow$ **R** is said to be a convex function if

$$f\left(\lambda x+(1-\lambda)y\right)\leq \lambda f(x)+(1-\lambda)f(y)$$

whenever x, y are in V and $\lambda\in[0,1]$.

The roots of the idea lie in the book [5], in which the generalized convexity may be defined eventually by "replacing the linear structure of the space by another type of structure and defining a notion of straight-line segment". These kinds of convexity properties are called segmental convexities. As examples, are the concepts of

- E-convexity defined by E. A. Youness [12] in $\mathbf{R}^n$, in which the straight line segment is $<x,y> \rightarrow \lambda E(x)+(1-\lambda)E(y)$, E: $\mathbf{R}^n\rightarrow\mathbf{R}^n$.
- g-convexity defined by M. A. Noor [10] in $\mathbf{R}^n$, in which the straight line segment is $<x,y> \rightarrow \lambda x+(1-\lambda)g(y)$, g: $\mathbf{R}^n \rightarrow \mathbf{R}^n$.
- order convexity defined by G. Birkhoff in 1948 (see [5]), which bases on the order segment.
- metric convexity defined by K. Menger in 1928 (see [5]), which bases on the order segment.
- average convexities induced by metric structures (see [5])

- convexity with respect to a set defined by L. Lupşa in 1980 (see [5])
- etc. see [5]

More convexity concepts defined in this manner imply properties as:

Theorem 1. *Let* (V, +, ·) *a real linear space and* f: V → **R** *a convex function. Then the level set* {x∈V | f(x) ≤ λ} *is convex for every* λ ∈ R.

Theorem 2. *Let* (V, +, ·) *a real linear space and* f: V → **R** *a concave function. Then the level set* {x∈V | f(x) ≥ λ} *is convex for every* λ ∈ **R**.

The main results of this paper refer to a touch screen method to approximate the solution of the vectorial optimization problem

$$(P):\ \begin{cases} F(x) = (f_1(x), f_2(x), \dots, f_n(x)) \to v-\max \\ x \in D \end{cases},$$

where D is a subset having some type of segmental convexity in a real linear space and each function f_k: D → **R** has the corresponding convexity for functions, k∈{1,2,…,n}.

By means of a classical reasoning we proved the following property:

Theorem 3. *Let* (V, +, ·) *a real linear space,* D⊆V *a segmentally convex set and* f: V → **R** *a function having the corresponding convexity property. Then the level set* {x∈V | f(x) ≤ λ} *is segmentally convex in the same manner as* D *for every* λ ∈ **R**.

If f: D → **R** is a function then the collection of level sets {x∈V | f(x) ≤ λ} for some λ∈K ⊆ **R** determines a representation of f by means of a map (see [2]).

In order to solve problem (P), we use the following procedure:

Step 1. Define a set of levels, K ⊆ **R**.
Step 2. For an element λ∈K, draw the level sets of each f_k, k∈{1,2,…,n}.
Step 3. Compute the intersection of the level sets.
Step 4. Compute the diameter of the intersection of the level sets.
Step 5. Compare the diameter of the intersection of the level sets with the desired error. If the diameter does not exceed the desired error, the every point of the intersection is a convenient approximation of the solution of (P). Go to step 6. Otherwise go to step 7.
Step 6. Choose a point from the intersection of the level sets and declare it as the solution of the problem.
Step 7. Diminish the value of λ∈K and go to step 2.

In order to compute the intersection of the generalized convex level sets we can use either the procedures from [1]. The diameters of the intersections of the convex level sets are computed following the procedures from [8].

If the level sets are not classically convex, but have a generalized convexity property, then we determine its digital convex hull as in [3] or as in [11] and follow the above described procedure.

Let us illustrate the procedure on an example of two dimensional g-convex optimization problem, according to the concepts studied in [6]. First we study this kind of convexity and then we conclude by solving a practical example.

Let **R** be the set of real numbers, and g: $\mathbf{R}^n \to \mathbf{R}^n$ be a transformation of $\mathbf{R}^n$, M ⊆ R^n.

Definition 4. *The set* M *is said to be g-convex if* M, *whenever*

$$(1 - \lambda)x + \lambda g(y) \in M,$$

for every x, y ∈ M *and* λ ∈ [0, 1].

Suppose that M is g-convex.

Definition 5. *A function* f : M → R *is said to be* g-*convex* (or g-*concave*) *if*

$$f((1 - \lambda)x + \lambda g(y)) \le (\text{or } \ge)\ (1 - \lambda)f(x) + \lambda f(g(y)),$$

for every x, y ∈ M *and* λ ∈ [0, 1].

The concepts of g-convex set and function are of preinvex type, with the particular additional pound η(x,y)=g(y) – x. These types of preinvexities functions are introduced by Jeyakumar [7] and Hanson & Mond [9].

Let us consider $M \subseteq \mathbf{R}^n$, M g-convex. For λ >0 let us denote by

$$f_\lambda(M) = \{x\in M|\ f(x) \le \lambda\} \text{ if } f \text{ is g-convex,}$$

$$f_\lambda(M) = \{x\in M|\ f(x) \ge \lambda\} \text{ if } f \text{ is g-concave.}$$

Property 6. *If f*: M → **R** *is* g-*convex* (g-*concave*) *and* f(g(x)) ≤ λ *whenever* f(x) ≤ λ *then the level set* $f_\lambda(M)$ *is* g-*convex.*

Theorem 7. *If f is a* g-*convex* (g-*concave*) *function then the set of all the dominated* (*non-dominated*) *points of f is given by the intersection of all its nonempty level sets.*

Let us solve the following problem:

Example 8. Let us consider the domain D defined as the astroide of parameter a = 4,

$$D = \left\{(x,y)\in R^2 \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} \le 4^{\frac{2}{3}}\right\}.$$

D is g-convex with respect to the grid set of step 4, when g is the function which associates the center point of a grid cell with all the points of the grid cell also containing its left and below sides. Let us consider the problem

$$(Q):\quad \begin{cases} F(x,y) = (f(x,y), h(x,y)) \to v - \max \\ (x,y) \in D \end{cases},$$

with the two g-concave components of F(x,y) defined by

$$f(x,y) = -\frac{1}{2}\left(x^2 + y^2 - 16\right),$$

$$h(x,y) = \begin{cases} -2x - 2\left(1-2\sqrt{2}\right)y + 8 & if \quad x\in[0,4]\ and\ 0 \le y \le x \\ -2y + 2\left(1-2\sqrt{2}\right)x + 8 & if \quad x\in[0,\sqrt{2}]\ and\ x \le y \\ -2x - 2\left(1-2\sqrt{2}\right)y + 8 & if \quad x\in[0,4]\ and\ -x \le y \le 0 \\ 2y + 2\left(1-2\sqrt{2}\right)x + 8 & if \quad x\in[0,\sqrt{2}]\ and\ y \le -x \\ -2y - 2\left(1-2\sqrt{2}\right)x + 8 & if \quad x\in[-\sqrt{2},0]\ and\ -x \le y \\ 2x + 2\left(1-2\sqrt{2}\right)y + 8 & if \quad x\in[-4,0]\ and\ 0 \le y \le -x \\ 2x - 2\left(1-2\sqrt{2}\right)y + 8 & if \quad x\in[-4,0]\ and\ x \le y \le 0 \\ +2y - 2\left(1-2\sqrt{2}\right)y + 8 & if \quad x\in[-\sqrt{2},0]\ and\ y \le x \end{cases}$$

The graphs of the two functions f and h are represented in figures 1 and 2.

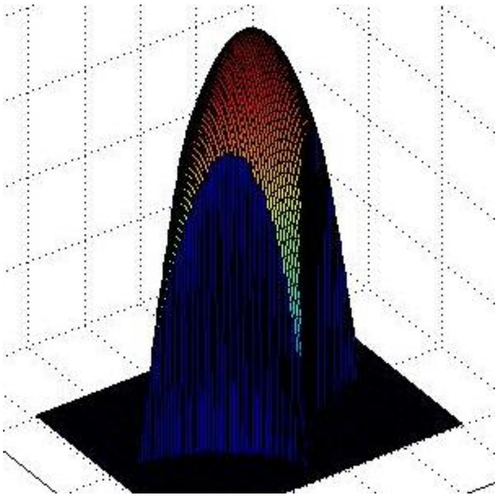

Figure 1. Graph of function f

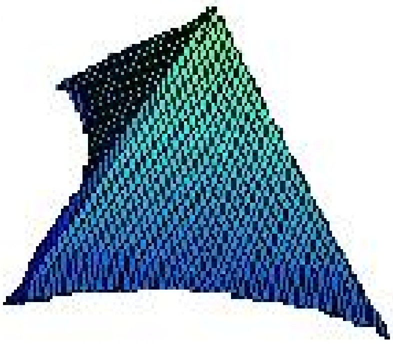

Figure 2. Graph of function h

It is easy to see that

$$\text{Max } f(x,y) = f(0,0) = 8$$
$$\text{Max } h(x,y) = h(0,0) = 8.$$

As consequence, the unique Pareto point of function F is (0,0) and the maximum value is

$$\text{v-max } F(x,y) = F(0,0) = (8,8).$$

Now, let us solve the problem by means of the above described procedure. The map-representations of the two functions using level sets situated at 0.5 units from each other are in figures 3 and 4.

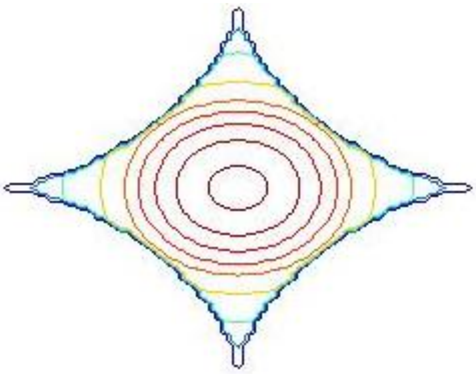

Figure 3. Map-representation of function f

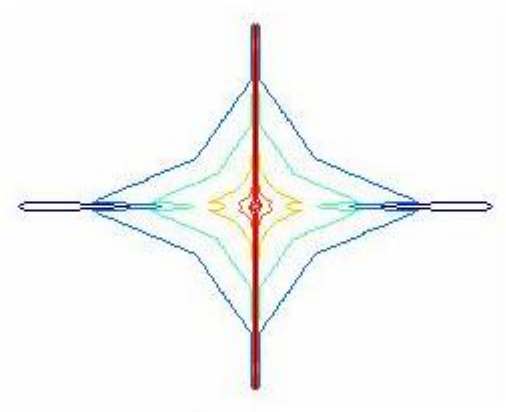

Figure 4. Map-representation of function h

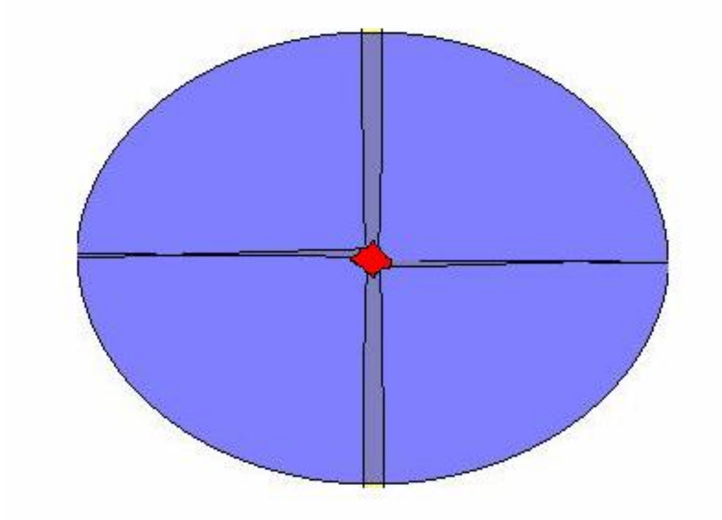

Figure 5. Intersection of the two level sets with λ=7.5

The intersection of the two level sets of λ = 7.5 is represented in figure 5. The level set of function f, for λ = 7.5 is a disk having the radius equal to 1. The level set of function h is a four vertices star included in this disc, having the vertices on the circle. Their intersection is the star itself. If we pick up a point situated in this star, then the error of the approximation of the Pareto point of coordinates (0,0) by the chosen point does not exceed the diameter of the intersection of the level sets, d = 2.

This procedure is easy to use by a touch screen technique, leading to a method of easy solving each vectorial optimization problem. It is necessary to have an existence theorem for the solution, which a generalized convexity property may guarantee.

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