CONTRIBUTION TO ANALYSIS OF CRITICAL LOAD OF THE SLENDER BEAMS AND THIN PLATES USING FINITE DIFFERENCE METHOD

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Abstract: Advanced design based on the different material of structures includes better use of materials, less weight compared to the equivalent isotropic construction and controlled effectively reserve resistance in all its segments.
In this case a calculation of critical load is exposed using the FDM (finite difference method) concepts of slender beams thin plates subjected to complex loads due to forces in the middle-plane.
Developed theoretical relations can be applied in the calculation of practically all structures or other structures that can be numerically modeled as beam and plate. Such examples are the spars, bars and shells of the aircraft, boat beams and panels, beams and plates in building construction and the similar objects.
The results, presented in this paper refer to the calculation of critical load of quite often models of beams and plates as they are encountered in structural analysis of typical structures, are given in graphic form.

1. INTRODUCTION

Starting from the basic ideas by authors to be reliable solution should always look for on the basis of many models, this will present some specific use FDM models, such as education and example for use in engineering practice. Reason is due to point to the advantage of using FDM for the identification of complex structures in relation to the idealized geometric forms and material favored isotropic structure, which allows to obtain elegant solutions at the level of equations and diagrams but essentially does not solve problems in practice on the required level. So it will be here to analyze the general models. For beams will primarily be confirmed results already exist in mechanics.
Structural analysis and optimization of construction, with orthotropic or hybrid-type plate structure, can be performed according to [1,8]. It is evident that the concept of structural analysis of the quasi-isotropic structure can not be applied, because it contents a bigger errors, in dependence of input of specific mechanical characteristics significant for calculate members of the matrix of elasticity and stiffness of the orthotropic structure like composite.

Orthotropic advantages over isotropic materials are otherwise well known. That can be controlled use in logical adequate computational models, suitable for carrying out structural analysis based on the respective applicable software.
In this case will be focused attention on the implementation FDM [1,3,4,8], which can be an useful reference for the calculation of parameters of different materials and geometric configuration of structures.

2. BASIC THEORETICAL RELATIONS

2.1. EQUATION OF BUCKLING OF SLENDER BEAMS

The basic differential equations for the problem of buckling beams [4], Figure 1, is given as (1).

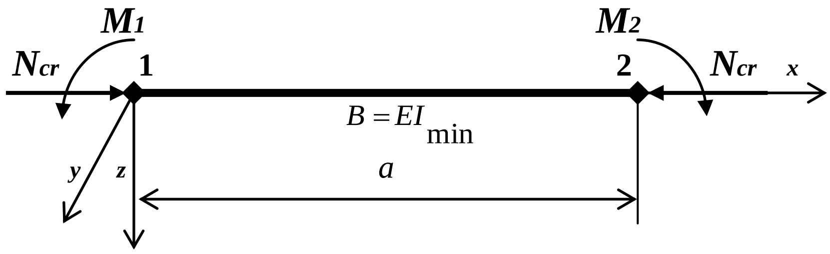


Figure 1. Beam loaded by N_{cr} force

$$\frac{\partial^2 W}{\partial X^2} = -BW \quad ; \qquad \textit{W-deflection} \tag{1}$$

For the boundary conditions, it can be assumed that edges of beam are clamped or simply-supported or any other combinations. Then will appear the following situations:

$$\left.\frac{\partial W}{\partial X}\right|_{x=0} = 0 \qquad \left.\frac{\partial W}{\partial X}\right|_{x=a} = 0$$
$$W(0) = W(a) = 0 \tag{2}$$

$$\left.\frac{\partial^2 W}{\partial^2 X}\right|_{x=0} = 0 \qquad \left.\frac{\partial W}{\partial X}\right|_{x=a} = 0$$
$$W(0) = W(a) = 0 \tag{3}$$

$$\left.\frac{\partial W}{\partial X}\right|_{x=0} = 0 \qquad W(0) = 0 \tag{4}$$

$$\left.\frac{\partial^2 W}{\partial^2 X}\right|_{x=0} = 0 \qquad \left.\frac{\partial^2 W}{\partial^2 X}\right|_{x=a} = 0$$
$$W(0) = W(a) = 0 \tag{5}$$

It is possible more variations of supports and the corresponding load (boundary conditions: clamped on both ends of the beam, clamped on one end and the second simple supported, one end is clamped and the other end is free, both ends are simple supported).

2.2. EQUATION OF THIN PLATE BUCKLING

It is evident, rational design is generally appeared of orthotropic shells and plates, and will be a concept of calculation of resistance orthotropic structure as focus of the problem of buckling, ie. it is need to find the value of critical forces for flat orthotropic plates (as one of the typical substructure of main structures), with high loads acting in-plane $\{N\} = \{N_x, N_y, N_{xy}\}^T$.

The basic differential equations of deformation (deflection) of flat plate, Figure 2, subject to deflection due to load **{**N**},** is given as (6), otherwise is discussed in [5,6,7,8,9].

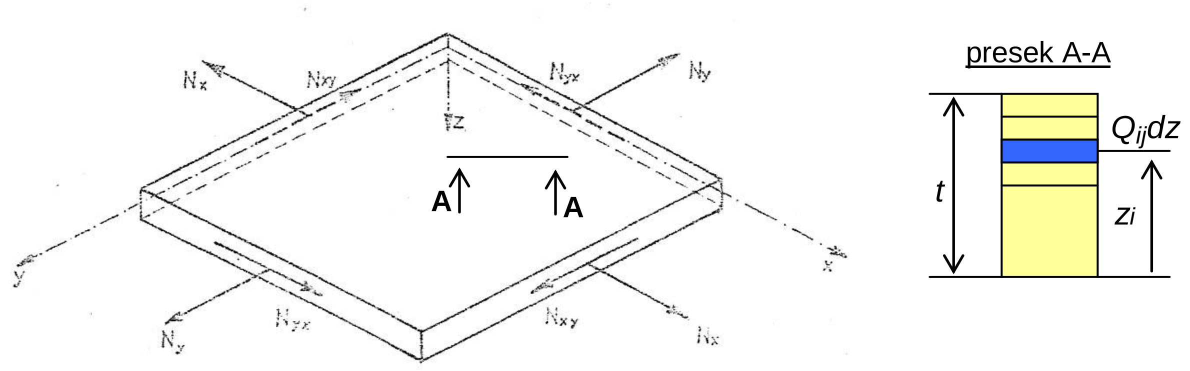


Figure 2. Layered orthotropic rectangular plate

$$D_{11}W_{xxxx} + 2(D_{12} + 2D_{66})W_{xxyy} + D_{22}W_{yyyy} + 4(D_{16}W_{xxxy} + D_{26}W_{xyyy}) = \\ = N_x W_{xx} + N_y W_{yy} + 2N_{xy}W_{xy} \quad (6)$$

where:

W - deflection of middle surface of plate,
N_x, N_y, N_{xy} - normal and shear forces per unit length dependent x and y coordinates,
D_{ij} - member of bending stiffness matrix of the composite laminate [1,2,5],

$$D_{ij} = \int_t \bar{Q}_{ij}\, Z^2 dz \,,$$

$\bar{Q}_{ij}$ - appropriate members of the reduced stiffness matrix [1,5].

In Figure 2, forces N_x, N_y, N_{xy} as load of rectangular plate are represented by single vectors. Main directions in Figure 2, were adopted as the positive directions of forces.

Evidently, when the plate divided (m x n) on the smaller rectangles, as shown in Figure 4, it is possible, via an operator [3], for each nodal point to achieve the corresponding equation from basic diferencne equation (10).

So it can get a system of homogeneous linear equations and solutions other than trivial (it is considered deformed plate), if the system determinant is zero. From this condition it will be possible to calculate values of critical forces.

For the boundary conditions, it can be assumed that all edges of plate are along clamped. But also can be considered a problem if the plate along two opposite sides are clamped and freely along the other two sides (or simply-supported) or any other combinations. Then will appear the following situations:

$$\left.\frac{\partial W}{\partial Y}\right|_{\substack{x=0 \\ x=a}} = 0 \qquad \left.\frac{\partial W}{\partial X}\right|_{\substack{y=0 \\ y=b}} = 0$$

$$W(0,Y) = W(a, Y) = W(X, 0) = W(X, b) = 0 \quad (7)$$

$$\left.\frac{\partial^2 W}{\partial Y^2}\right|_{\substack{x=0 \\ x=a}} = M1^{*} \qquad \left.\frac{\partial W}{\partial X}\right|_{\substack{y=0 \\ y=b}} = 0$$

$$W(0,Y) = W(a, Y) = W(X, 0) = W(X, b) = 0 \quad (8)$$

$$\frac{\partial^2 W}{\partial X^2}\Bigg|_{y=b}^{y=0} = M_2^* \qquad \frac{\partial W}{\partial Y}\Bigg|_{x=a}^{x=0} = 0$$

$$W(0,Y) = W(a, Y) = W(X, 0) = W(X, b) = 0 \tag{9}$$

In any case, the boundary conditions must be always accurately taken into account, otherwise you can get the wrong result, which does not match the real of problems.
In the case of isotropic structure, equation (6) is reduced to (10), and the boundary conditions defined in the (8) and (9) could be expressed as (11) and (12).

$$\frac{\partial^4 W}{\partial X^4} + 2\frac{\partial^4 W}{\partial X^2 \partial Y^2} + \frac{\partial^4 W}{\partial Y^4} = \frac{1}{D_{11}}\left(N_x \frac{\partial^2 W}{\partial X^2} + N_y \frac{\partial^2 W}{\partial Y^2} + 2N_{xy}\frac{\partial^2 W}{\partial X \partial Y} \right) \tag{10}$$

$$\frac{\partial^2 W}{\partial Y^2}\Bigg|_{x=a}^{x=0} = 0 \qquad \frac{\partial W}{\partial X}\Bigg|_{y=b}^{y=0} = 0$$

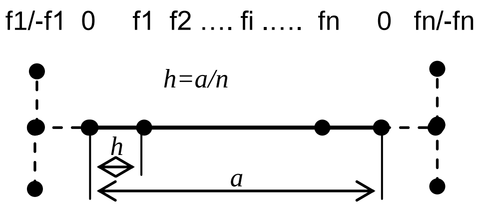
$$W(0,Y) = W(a, Y) = W(X, 0) = W(X, b) = 0 \tag{11}$$

$$\frac{\partial^2 W}{\partial X^2}\Bigg|_{y=b}^{y=0} = 0 \qquad \frac{\partial W}{\partial Y}\Bigg|_{x=a}^{x=0} = 0$$

$$W(0,Y) = W(a, Y) = W(X, 0) = W(X, b) = 0 \tag{12}$$

3. ADAPTATION OF EQUATIONS WITH FDM APPLICATION

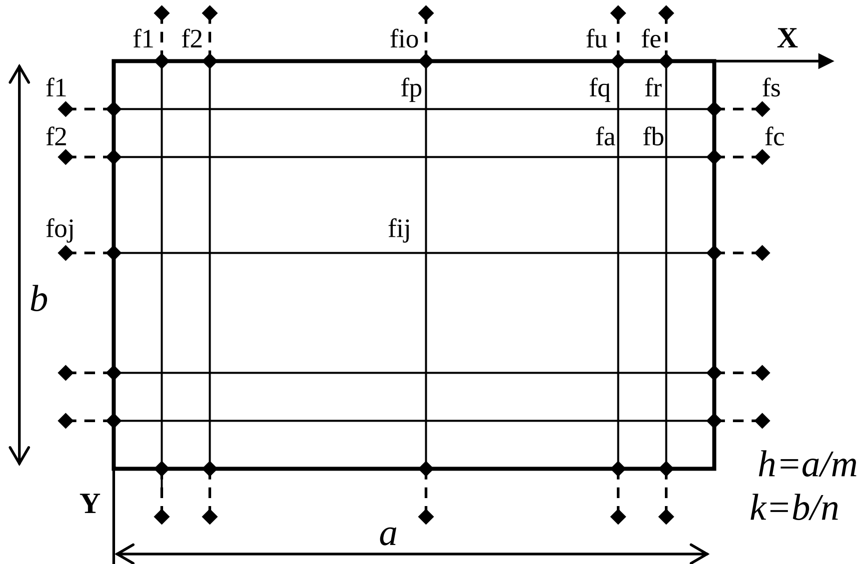
For the models in Figure 3 and 4, the basic differential equation (1) and (6), using the FDM on the basis of [3], and stiffness matrix according to [2,5,8], becomes (13) and (14).



fij - values of deflection outside the contours correspond values fij, for the case of clamped*
-fij- values of deflection outside the contours correspond values fij, for the case of simply-supported*

Figure 3. Calculation model-discretization of beam

$$B\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} = -N_x f_i \tag{13}$$



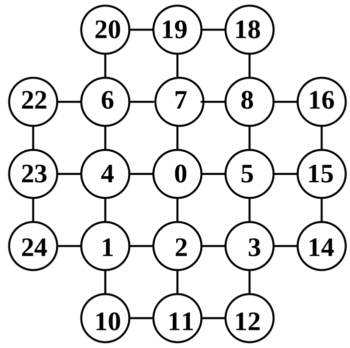


Figure 4. Calculation model-discretization of plate

$$\left(\frac{6D_{11}}{h^4}+\frac{8D_{12}+16D_{66}}{h^2k^2}+\frac{6D_{22}}{k^4}+\frac{2N_x}{h^2}+\frac{2N_y}{k^2}\right)f_o+\left(\frac{2D_{12}+4D_{66}}{h^2k^2}-\frac{2D_{16}}{h^3k}-\frac{2D_{26}}{hk^3}+\frac{1}{2}\frac{N_{xy}}{hk}\right)f_1+$$
$$+\left(\frac{-4D_{12}-8D_{66}}{h^2k^2}-\frac{4D_{22}}{k^4}-\frac{N_y}{k^2}\right)f_2+\left(\frac{2D_{12}+4D_{66}}{h^2k^2}+\frac{2D_{16}}{h^3k}+\frac{2D_{26}}{hk^3}-\frac{1}{2}\frac{N_{xy}}{hk}\right)f_3+$$
$$+\left(-\frac{4D_{11}}{h^4}-\frac{4D_{12}+8D_{66}}{h^2k^2}-\frac{N_x}{h^2}\right)f_4+\left(-\frac{4D_{11}}{h^4}-\frac{4D_{12}+8D_{66}}{h^2k^2}-\frac{N_x}{h^2}\right)f_5+$$
$$+\left(\frac{2D_{12}+4D_{66}}{h^2k^2}+\frac{2D_{16}}{h^3k}+\frac{2D_{26}}{hk^3}-\frac{1}{2}\frac{N_{xy}}{hk}\right)f_6+\left(\frac{-4D_{12}-8D_{66}}{h^2k^2}-\frac{4D_{22}}{k^4}-\frac{N_y}{k^2}\right)f_7+$$
$$+\left(\frac{2D_{12}+4D_{66}}{h^2k^2}-\frac{2D_{16}}{h^3k}-\frac{2D_{26}}{hk^3}+\frac{1}{2}\frac{N_{xy}}{hk}\right)f_8+\left(\frac{D_{26}}{hk^3}\right)f_{10}+\left(\frac{D_{22}}{k^4}\right)f_{11}-\left(\frac{D_{26}}{hk^3}\right)f_{12}+$$
$$-\left(\frac{D_{16}}{h^3k}\right)f_{14}+\left(\frac{D_{11}}{h^4}\right)f_{15}+\left(\frac{D_{16}}{h^3k}\right)f_{16}+\left(\frac{D_{26}}{hk^3}\right)f_{18}+\left(\frac{D_{22}}{k^4}\right)f_{19}-\left(\frac{D_{26}}{hk^3}\right)f_{20}-\left(\frac{D_{16}}{h^3k}\right)f_{22}+$$
$$+\left(\frac{D_{11}}{h^4}\right)f_{23}+\left(\frac{D_{16}}{h^3k}\right)f_{24}=0\ ;\quad W_i=f_i \tag{14}$$

where: $h = a/m$ - the finite segment/step-by-axis x;
$k = b/n$ - the finite segment/step-by-axis y.

If the coefficients with the displacements from equation (14) notation as I, J ,..., N, then it can perform a clear model to identify the coefficients for generalized orthotropic, orthotropic and isotropic plates starting from the generalized model of orthotropic plates, equation (14). In relation to previous findings, see Figure 5, the closer identification is given in [8].

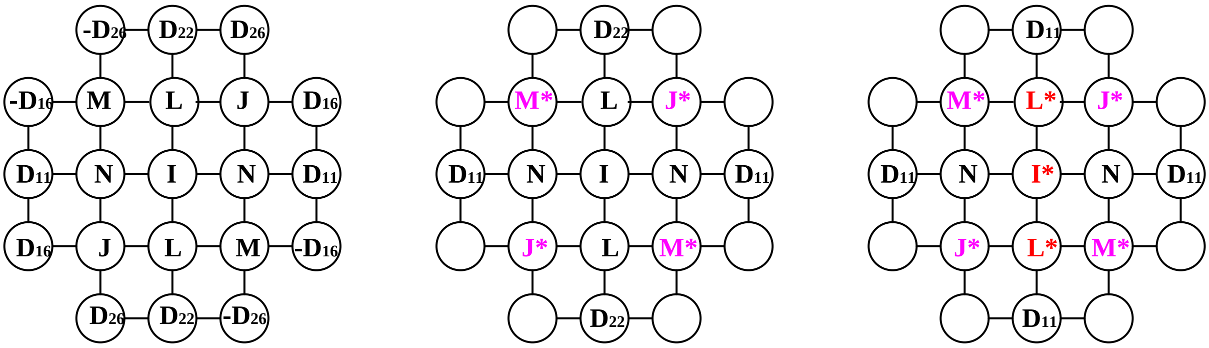


Figure 5. Schematic representation of the disposition coefficients for the appropriate node displacements of plates of different material configurations

The present model is particularly suitable when we want to check whether the vector of work load - $\{N_r\}$ is a load greater than the critical or not.

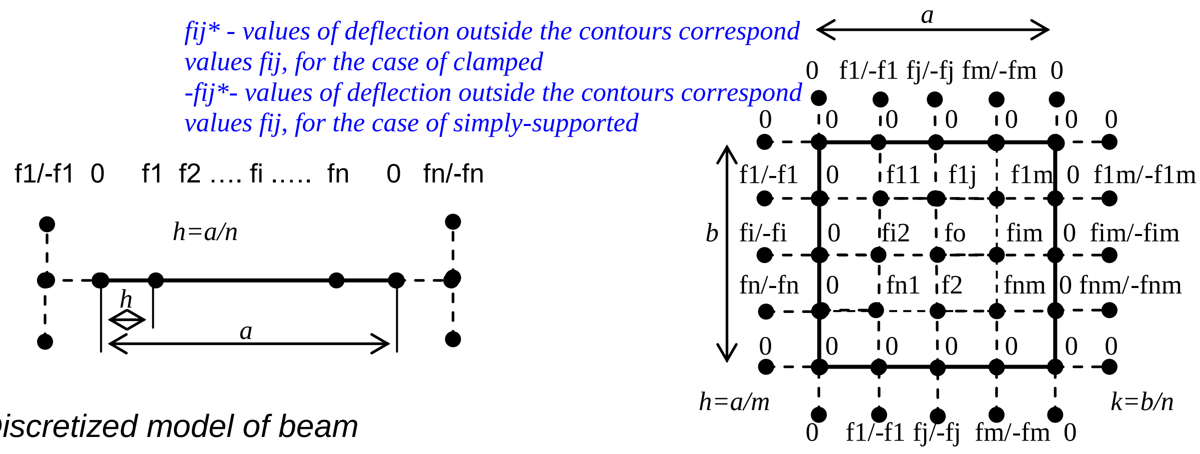
Then, for the operating stresses $\{N_r\}$ which are calculated for the applicable load, can identify factors λ_f (eg. $\lambda_f = 1 \div 20$, which means that the incremental raising of load level from 5% to 100% of the actual load is done), so if there is no change of determinant sign of a system of linear algebraic equations means that there is no (at least the first mode) buckling, ie. can be concluded that the workload of less than critical one [8] and in the case of linear analysis possible to determine whether the specific combination of forces (as a result of operating load) on plate causes the level of the critical situation which refers buckling of plate or not.

$$\{N_{cr}\} = (\lambda_f)\begin{Bmatrix} \bar{N}_x \\ \bar{N}_y \\ \bar{N}_{xy} \end{Bmatrix} \tag{15}$$

Such as this possibility, other concepts are not provided that, and author believes the particular concern of this model could be considered as one of the most appropriate underlying model for education and training.

4. RESULTS OF THE CALCULATION MODEL

The idealized and discretized structures of the beam and plate are given in Figure 6. Basic material properties are given in accordance with equations (16), (18) and (19).



a) Discretized model of beam

b) Discretized model of plate

Figure 6. Basic FDM models for calculation of beam and plate

In the event that it is a composite, ie. the generalized orthotropic material, then the material properties are calculated from equation (17) to (19), in accordance with [5,8,9]. Otherwise, the isotropic structure of the respective characteristics of interest to form a matrix of elasticity Qij, obtained in accordance with [2,4].

$$\left[\overline{Q}\right] = [T]^{-1}[Q][T]^{-T} \tag{16}$$

$$[Q] = \begin{bmatrix} 13.892 & 305 & 0 \\ 305 & 1017 & 0 \\ 0 & 0 & 430 \end{bmatrix} \tag{17}$$

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta\sin^2\theta \end{bmatrix} \tag{18}$$

$$\left[\overline{Q}\right] = \begin{bmatrix} 8.315 & 2.664 & 4.149 \\ 2.664 & 1.878 & 1.426 \\ 4.149 & 1.426 & 2.788 \end{bmatrix} \tag{19}$$

Appropriate stiffness for the beam and plate are given as (20).

$$B = E_x \frac{bt^3}{12} \ \text{- for beam;} \qquad D_{ij} = \int_t \overline{Q}_{ij} Z^2 dz \ \text{- for plate} \tag{20}$$

Load, as in Figure 1 and 2, in this case load is given only for the axial compression in the direction of the axis "x", - Nx [dN/m], other forces are equal to zero.
This example is provided as an educational model, anyway, based on content from point 3, it is clear that can be applied most complex load vector.

The results of calculation are given in the form of the relative relationship of the obtained critical force for the appropriate number of nodes, according to Nx,cr (tentatively taken as the exact value), see Figure 7 and 8.

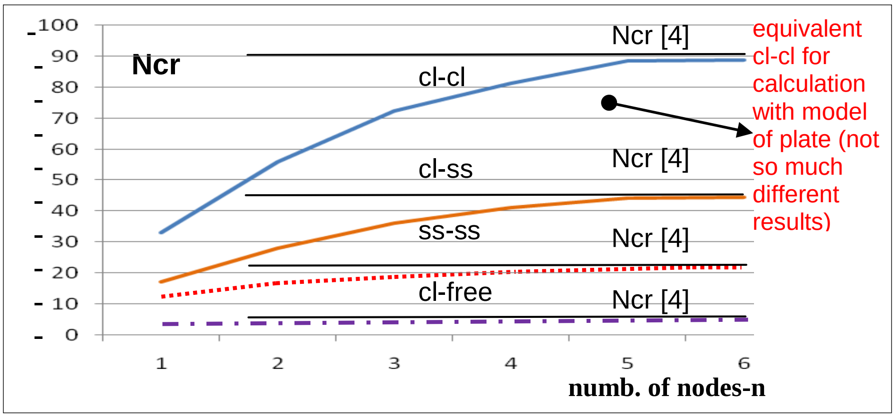


Figure 7. Results for beam

It is obvious that the rapid convergence of solutions to the exact value (it is best to get the experimental measurements of real model) if you go to the application of FDM and increases the number of nodes.

For beams, the results of acting in accordance with the expected theoretical principles. For plates, using FDM, which is presented here, the situation is somewhat different.

Errors, on the FDM results from relevant existed theory of resistance of materials [2,4] taken as accurate, are within an acceptable range for engineering practice.

The considerations presented here, in the case of beams clamped at both ends of the beam (cl-cl option) relative error of FDM concept, for example for the five nodes, is ε<|11,6%|, while for ss-ss version with support on both ends of the beam is ε<|3,1%|. Error for the cl-ss variant is ε<|5,3%|. The largest relative error for the cl-free variant is ε<|20%|. Of course, with an increase in the number of nodes, ie. reducing the step-size of increments, the relative error-ε would be significantly reduced.

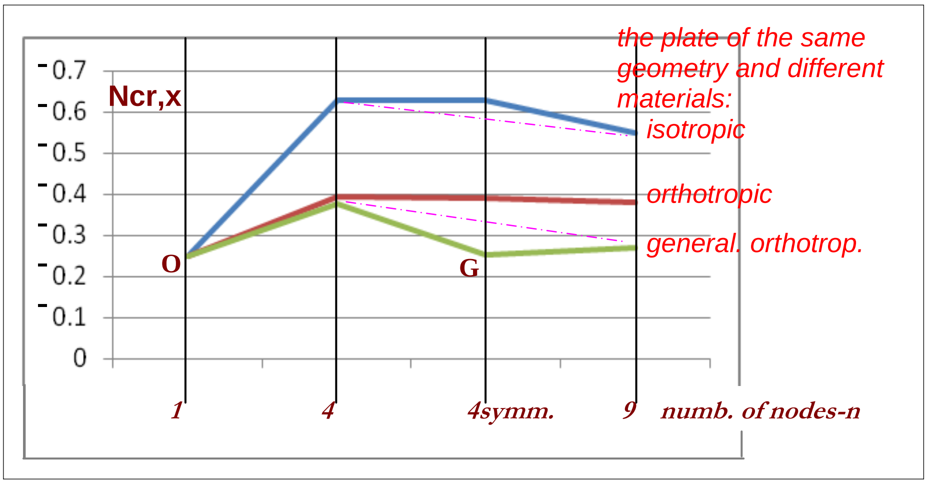


Figure 8. Results for plate

In Figure 8, we can see the anomalies of results in the points O and G. At point O it is clear that this is due to lack of a dense network of nodes, otherwise in both cases (O,G) does not capture the mechanisms of the general orthotropic material, because we have the symmetrical model of set of points for referring to discrete values of deflection (the model is obviously asymmetric). It is the best to increase a number of nodes (denser network) and to apply the general model of expected displacement-deflection. Then it will decrease the error on the results of calculations. In calculation of this plate, relative errors for plate with different variant with support are within the range of ε= 2 to 16%, but just for a network of 9 (3x3) nodes.

It should be noted however that obtained results based on FDM give the satisfactory level of accuracy, especially for a reasonable increase in the number of nodes or elements.

Convenience implementation of FDM especially comes into play when you have just analysis of the critical stage of plate, as can be seen in comparative references [7,9], where they are used here difference base of developed equations, with additional members of the force in the secondary level.

5. CONCLUSION

One of the conclusions is that with satisfactory accuracy can be accepted model calculations using FDM for the beam, with the requirement to elect a relatively dense network of nodes.

Evidently, when the beam discretized in smaller segments (h) and the plate discredized (m x n) in smaller rectangles (h,k), as shown in Figure 6, it is possible through the appropriate operator [3] for each nodal point to get the equations based on the corresponded difference equations (13) and (14).

So you get a system of homogeneous linear equations, for which there will be solutions other than trivial (it is considered deformed plate), if the system determinant is zero. From this condition the value of critical force is calculated.

The boundary conditions are concerned, it can be assumed that all edges of plate along are clamped, but can also be considered a problem if the panel along two opposite sides are clamped and freely along the other two sides (or simply-supported) or any other combination.

Based on the conducted analysis it can be concluded that the rapid convergence of solutions to the exact value (it is best to get the experimental measurements of real model) if you go to the application of FDM and increases the number of nodes.

It should be noted however that the FDM obtained through the satisfactory level of accuracy of results, especially for a reasonable increase in the number of points or elements.

Presented results should serve as an indicator of the expansion of theoretical base of similar models, which can be reasonably use by researchers and engineers in their practices, and by students for educational purposes.

Finally we must point out that it is presented the quality education examples for students and professionals who perform structural analysis of structures of different material configurations.

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