ANNALS of the ORADEA UNIVERSITY.
Fascicle of Management and Technological Engineering, Volume X (XX), 2011, NR2

MUTUALLY CONJUGATED TRIGONOMETRIES FOR THE HILL'S EQUATION

Stana Cvejic, Milena Lekic
Faculty of Natural Sciences and Mathematics,
University in Kosovska Mitrovica, Serbia mvasiljevic1@open.telekom.rs

Abstract: The oscillating solutions of some differential equation can be declared trigonometric if a minimum of the analogy with the Euclid's trigonometry is fulfilled, such as follows:

- if it is possible to form a triangle from solutions, the same as from the length;
- if it is possible for some elements of the solutions to be presented as real angles in the plain surface or space;
- relation which connects the basic (fundamental) solutions of the differential equation with oscillating solutions should exist;
- there should be possibility for all solutions of the equation to be expressed as linear combination of the fundamental solution;
- there should exist the equivalent of the Oiler and Moivre's formulae.

If all above mentioned conditions are present, then as a consequence there are analogical theorems, formulae for double angles and semi-angles, formulae of the sine, cosine and tangent theorems, trigonometric relations and derived trigonometric functions. In this work, we have shown that the oscillating solutions of the Hill's equation fulfill the conditions which we have mentioned and make the trigonometry T_g $(\sin_{a(x)} x, \cos_{a(x)} x)$. Then we have determined the standardized trigonometry T_g^* (cosx*, sinx*) of the new argument x^* which is conjugated trigonometry of the Hill equation's trigonometry.

Key words: Differential equations, Hill's equation, trigonometries, conjugated trigonometries, periodicity, oscillatority.

1. HILL'S EQUATION AND EQUIVALENT ANGLES FOR ITS OSCILLATING SOLUTIONS

Regarding the classic Kneser's theorem (If *a(x)>0* and if integral $\int_0^{+\infty} a(x)dx$ diverges, all solutions of the canonic equation of the second order are oscillating), the Hill's equation

(1) $y'' + a(x) \cdot y = 0$, *a(x)>0*

define many different trigonometries.

We have shown that solutions of the equation (1), given with sequences

(2) $$\begin{cases} y_1 = 1 - \int_0^x \int_0^x a(x)dx^2 + \int_0^x \int_0^x a(x)dx^2 \int_0^x \int_0^x a(x)dx^2 - ... = \cos_{a(x)} x, \\ y_2 = x - \int_0^x \int_0^x xa(x)dx^2 + \int_0^x \int_0^x a(x)dx^2 \int_0^x \int_0^x xa(x)dx^2 - ... = \sin_{a(x)} x \end{cases}$$

which converge for every interruptible *a(x)* in every canonic interval, are oscillating with zeros and periods which depend on *x*. They are also not limited with one.

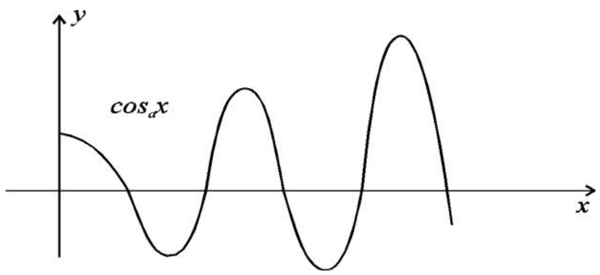

Figure 1.

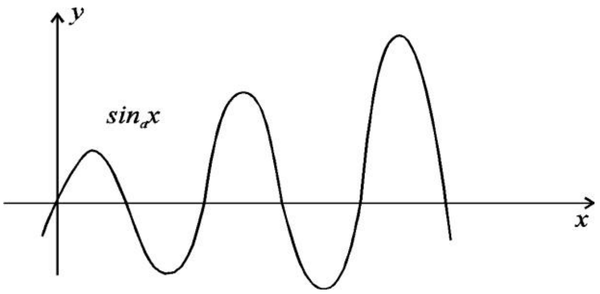

Figure 2.

Thus, different trigonometries are obtained in the following forms:

- classic Euclid's trigonometry

$$y'' + 1 \cdot y = 0 \qquad \text{and} \qquad y'' + \lambda^2 y = 0$$

$$y_1 = \cos x, \; y_2 = \sin x \qquad y_1 = \cos \lambda x, \; y_2 = \sin \lambda x$$

$$T = 2\pi \qquad T = \frac{2\pi}{\lambda}$$

- trigonometry of the Mathieu's functions
- trigonometry of the Lame's functions
- trigonometry of the Bessel's functions, etc

It is easy here to obtain the basic trigonometric relation and derived trigonometric functions $tg_{a(x)} x$,
$ctg_{a(x)} x$. However, it is difficult to obtain Oiler's formula, because the functions (2) are not limited, while the Oiler's formula is extended within the unit circle. That is why the module of the function (2) should be limited in some way. In order to achieve this, we have introduced a real equivalent angle *x**, which has been defined by following relations:

$$(3) \qquad \begin{cases} \cos^2 x^* = y_1 \cdot y_2' \\ \sin^2 x^* = -y_1' \cdot y_2 \end{cases}$$

where y_1 and y_2 have been given with (2), while their derivatives have been given with our formulae:

$$(4) \qquad \begin{cases} y_1' = -\int_0^x a(x) \cdot y_1(x) dx = -\int_0^x a(x) \cdot \cos_{a(x)} x \cdot dx, \\ y_2' = 1 - \int_0^x a(x) \cdot y_2(x) dx = 1 - \int_0^x a(x) \cdot \sin_{a(x)} x \cdot dx. \end{cases}$$

The Wronskian of the equation (1) is

$$W(x) = W(y_1, y_2) = W(x_0) \cdot e^{-\int_{x_0}^{x} b(x) \cdot dx} = W(x_0),$$

where x_0 is a random point from the interval $[x_0, x]$, so we obtain the following:

$$W(x) = W(x_0) = W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = 1,$$

i.e.

$$(5) \qquad W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \cdot y_2' - y_1' \cdot y_2 = 1$$

The above relation also represent the basic trigonometric relation of the following trigonometry:

(6) $$y_1 \cdot (1 - \int_0^x a(x) y_2 \cdot dx) + (-\int_0^x a(x) y_1 \cdot dx) \cdot y_2 = 1$$

However, the formula (3) leads to the following:

(7) $$\begin{cases} \cos x^* = \sqrt{y_1 \cdot y_2'} \\ \sin x^* = \sqrt{-y_1' \cdot y_2} \end{cases}$$

Based on this we can state the following:

1. Trigonometric functions $\cos x^*$ and $\sin x^*$ do not have to be defined everywhere, because it is obvious that they must be valid for the relations $y_1 \cdot y_2' \geq 0$ and $-y_1' \cdot y_2 \geq 0$;
2. These functions also do not have to be limited by one, similarly as functions y_1 and y_2

In spite of these deficiencies, relations (7) can be accepted, because where the relations (7) are real and

Limited, the following is valid:

(8) $$\cos^2 x^* + \sin^2 x^* = y_1 \cdot y_2' - y_1' y_2 = 1.$$

Therefore, the basic trigonometric relation (still undetermined) of the trigonometry (7) is valid in the exactly standardized form of the classic trigonometry. The size *x** given with relation

(9) $$x^*(x) = arc\ \cos\sqrt{y_1 \cdot y_2'}$$

is real and can be considered to be an angle.

The normal conditions for the Hill's equation (1)

$$y_1(0) = 1,\ y_1'(0) = 0$$
$$y_2(0) = 0,\ y_2'(0) = 1$$

give $x^*(0) = arc\ \cos 1 = 0$ also from the derivative:

$$\left(x^*(x)\right)' = -\frac{1}{\sqrt{1 - y_1 \cdot y_2'}} \cdot (y_1' \cdot y_2' + y_1 \cdot y_2'') = -\frac{1}{\sqrt{1 - y_1 \cdot y_2}'} \cdot (y_1' y_2' - a(x) y_1 y_2)$$

because y_2 is the solution of (1), so it identically satisfies $y_2'' + a(x) \cdot y_2 \equiv 0$ and $y'' \equiv -a(x) y_2$. Thus, by applying L'Hopital's rule, we obtain $\left(x^*(x)\right)'_{x=0} = \frac{1}{2} a'(0)$. The graph of the function (9) until the first interruption has been given for $a(x) = e^x$ on the figure 3.

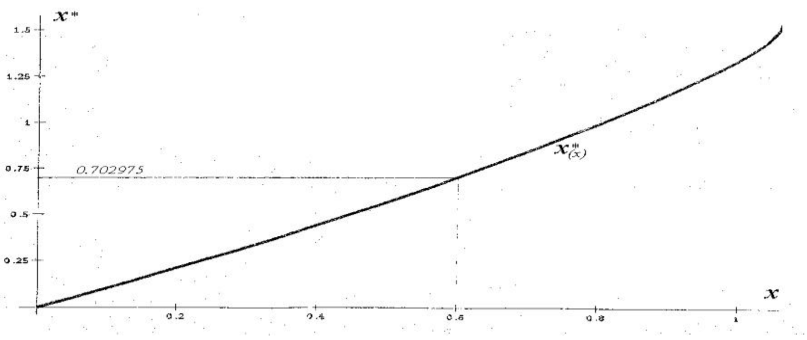

Figure 3.

Regarding the possible undefined areas of the functions (7) and (9) (see the next figure) in somewhat broader interval for *x*: *(0; 2 , 5)*, the graph of the connection between the angles *x** and *x* looks as presented on the figure 4 for the equation $y'' + e^x \cdot y = 0,$ which is also very important equation in the physics.

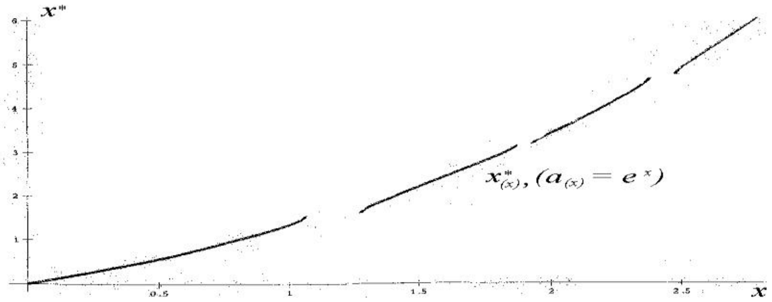

Figure 4.

For the same canonic equation for which $a(x) = e^x$, the graph of the function $\cos^2 x*$, depending on argument *x,* is presented on the following figure, where the intervals, on which the root $\sqrt{y_1 \cdot y_2{}'}$ is not defined or where it passes the amplitude one. These are interruptible distances on the figure 5.

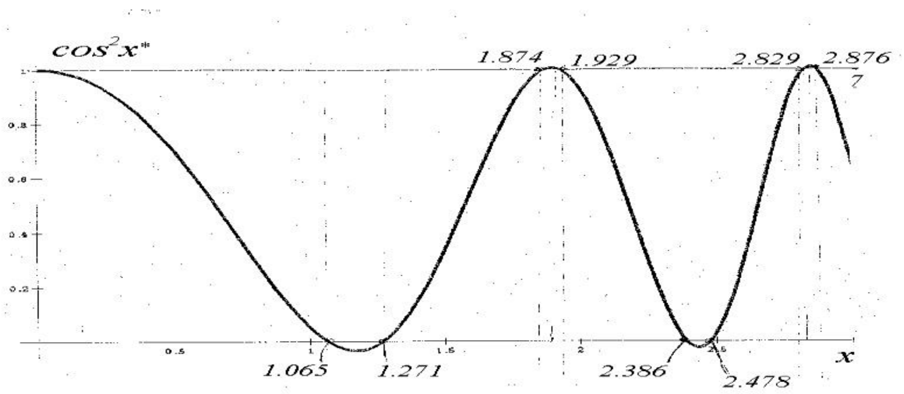

Figure 5.

If we research dependency of elements which influence $\cos x*(x)$, i.e. y_1' and y_2 and take the phase plain surface y_1' 0 y_2, then, regarding the interruptions, $y_2(y_1')$ for the same equation $y'' + a(x)y = 0$, $a(x) = e^x$, we shall obtain the following phase graph:

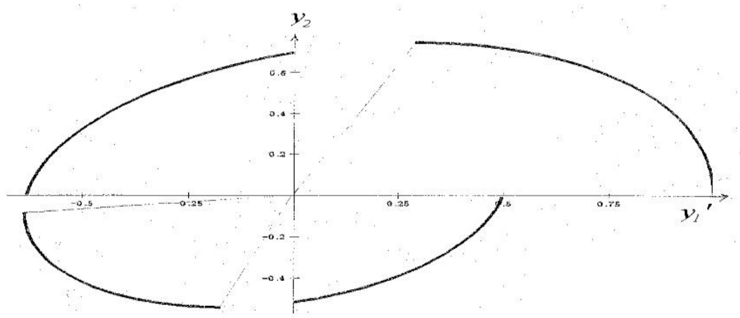

Figure 6.

By making such a choice of the formula (7), now for *x*(x)*, the Oiler's formula is valid.

(10) $\cos x^* + i\sin x^* = e^{ix^*} = \sqrt{y_1 y_2'} + i\sqrt{-y_1' y_2}$.

In this way the trigonometry of the Hill's equation is completed.

2. MUTUALLY CONJUGATED TRIGONOMETRIES

However, by this procedure we obviously form and connect two trigonometries:
1. Trigonometry of the solution of the Hill's equation (1)

(11) $Tg(y_1 y_2) = Tg(\cos_{a(x)} x, \sin_{a(x)} x)$ and

2. Standardized trigonometry of the new argument *x** given with the equations (7)

(12) $Tg^*(\cos x^*, \sin x^*)$

which are connected by formulae (1) –(10).
Now there is the following question: is the set of function s (12) also a trigonometry?
This question can be answered most easily if we form differential equation whose solutions are functions (7). For that purpose let us mark with:

(13) $$\begin{cases} Z_1 = \cos x^*(x) = \sqrt{y_1 \cdot y_2'} \\ Z_2 = \sin x^*(x) = \sqrt{-y_1' \cdot y_2} \end{cases}$$

First of all, we are surely interested whether the function Z_1 and Z_2 are mutually linearly independent. Since their Wronskian depends on the derivatives:

$$Z_1'(x) = (\cos x^*)_x' = \frac{1}{2\sqrt{y_1 y_2'}} \cdot (y_1' y_2' + y_1 y_2'') = \frac{y_1' y_2' - a(x) y_1 y_2}{2\sqrt{y_1 y_2'}}$$

$$Z_1''(x) = (\cos x^*)_{xx}'' = -\frac{1}{4} \cdot \left(y_1 y_2'\right)^{-\frac{3}{2}} \cdot (y_1' y_2' - a(x) y_1 y_2)^2 +$$

$$+\frac{1}{2}(y_1 y_2')^{-\frac{1}{2}} \cdot (y_1'' y_2' + y_1' y_2'' - a'(x) y_1 y_2 - a(x) y_1' y_2 - a(x) y_1 y_2')$$

$$Z_2'(x) = (\sin x^*)_x' = \frac{1}{2}(-y_1' y_2)^{-\frac{1}{2}} \cdot (a(x) y_1 y_2 - y_1' y_2')$$

$$Z_1''(x) = (\sin x^*)_{xx}'' = -\frac{1}{4} \cdot \left(-y_1' y_2\right)^{-\frac{3}{2}} \cdot (a(x) y_1 y_2 - y_1' y_2')^2 +$$

$$+\frac{1}{2}(y_1' y_2)^{-\frac{1}{2}} \cdot (-y_1'' y_2' - y_1' y_2'' + a'(x) y_1 y_2 + a(x) y_1' y_2 + a(x) y_1 y_2')$$

then it amounts to:

(14) $$W(Z_1, Z_2) = \begin{vmatrix} Z_1 & Z_2 \\ Z_1' & Z_2' \end{vmatrix} = \begin{vmatrix} \sqrt{y_1 y_2'} & \sqrt{-y_1' y_2} \\ \dfrac{y_1' y_2' + y_1 y_2''}{2\sqrt{y_1 y_2'}} & \dfrac{-y_1'' \cdot y_2 - y_1' y_2'}{2\sqrt{-y_1' \cdot y_2}} \end{vmatrix}$$

However, after the third calculation, we obtain:

(15) $$W(x) = \frac{a(x) y_1 y_2 - y_1' y_2'}{2\sqrt{-y_1 y_1' y_2 y_2'}}$$

From the relation (15), there follows that $W \neq const.$ and that *W* can have interruptions for $y_1 = 0,\ y_1' = 0,\ y_2 = 0$ and $y_2' = 0$, as well as whole intervals not defined, which leads to conclusion that trigonometry T_g* deviates from T_g. Nevertheless, most frequently $W \neq 0$ and functions (13) are linearly independent.

Now, we ask ourselves if there is linear homogenous differential equation of the second order for which (Z_1, Z_2) are fundamental particular integrals. In compliance with the theory, it states:

(16) $$\begin{vmatrix} Z'' & Z' & Z \\ Z_1'' & Z_1' & Z_1 \\ Z_2'' & Z_2' & Z_2 \end{vmatrix} = 0$$

or , if we develop determinant

$$Z''\begin{vmatrix} Z_1' & Z_1 \\ Z_2' & Z_2 \end{vmatrix} - Z'\begin{vmatrix} Z_1'' & Z_1 \\ Z_2'' & Z_2 \end{vmatrix} + Z\begin{vmatrix} Z_1'' & Z_1' \\ Z_2'' & Z_2' \end{vmatrix} = 0$$

or

$$W(x)\cdot Z'' - \begin{vmatrix} Z_1'' & Z_1' \\ Z_2'' & Z_2' \end{vmatrix}\cdot Z' + \begin{vmatrix} Z_1'' Z_1' \\ Z_2'' Z_2' \end{vmatrix}\cdot Z = 0$$

And since $W \neq 0$, we finally obtain:

(17) $$Z'' - \frac{\begin{vmatrix} Z_1'' Z_1 \\ Z_2'' Z_2 \end{vmatrix}}{W(x)}\cdot Z' + \frac{\begin{vmatrix} Z_1'' Z_1' \\ Z_2'' Z_2' \end{vmatrix}}{W(x)}\cdot Z = 0$$

or shorter

(18) $$Z'' + B(x)\cdot Z' + A(x)\cdot Z = 0$$

When in determinant (16) we substitute derivatives (14) and instead of coefficient *a=a(x)* we write only *a*, we obtain:

$$\begin{vmatrix} Z'' & Z' & Z \\ -\frac{1}{4}(y_1y_2')^{-\frac{3}{2}}\cdot(y_1'y_2' - ay_1y_2)^2 + \frac{1}{2}(y_1y_2')^{-\frac{1}{2}}\cdot(y_1'y_2' - ay_1y_2)' & \frac{1}{2}\cdot\frac{y_1'y_2' - ay_1y_2}{(y_1y_2')^{1/2}} & \sqrt{y_1y_2} \\ -\frac{1}{4}(-y_1'y_2' + ay_1y_2)^2 + \frac{1}{2}(-y_1'y_2' + ay_1y_2)' & \frac{1}{2}(-y_1'y_2' + ay_1y_2) & -y_1'y_2 \end{vmatrix} = 0$$

If the second row is multiplied by $\sqrt{y_1y_2'}$ and the third with $\sqrt{-y_1'y_2}$, we obtain:

$$\begin{vmatrix} Z'' & Z' & Z \\ -\frac{1}{4}\cdot\frac{(y_1'y_2' - ay_1y_2)^2}{y_1y_2} + \frac{1}{2}(y_1'y_2' - ay_1y_2)' & \frac{1}{2}\cdot(y_1'y_2' - ay_1y_2) & y_1y_2 \\ -\frac{1}{4}\cdot\frac{(-y_1'y_2' + ay_1y_2)^2}{-y_1'y_2} + \frac{1}{2}(-y_1'y_2' + ay_1y_2)' & \frac{1}{2}(-y_1'y_2' + ay_1y_2) & -y_1'y_2 \end{vmatrix} = 0$$

Now, in the obtained determinant we add the third row to the second one:

$$\begin{vmatrix} Z'' & Z' & Z \\ -\frac{1}{4}\cdot(y_1'y_2'-ay_1y_2)^2\cdot\left(\frac{1}{y_1y_2'}-\frac{1}{y_1'y_2}\right) & 0 & 1 \\ -\frac{1}{4}\cdot\frac{(-y_1'y_2'+ay_1y_2)^2}{-y_1'y_2}+\frac{1}{2}(-y_1'y_2'+ay_1y_2)' & \frac{1}{2}\left(-y_1'y_2'+ay_1y_2\right) & -y_1'y_2 \end{vmatrix}=0$$

If we develop determinant with respect to the elements of the first column, after short calculation, we obtain the following equation:

(19) $$Z''+\left[-\frac{(y_1'y_2'-a(x)y_1y_2)\cdot(y_1y_2'+y_1'y_2)}{2y_1y_1'y_2y_2'}+\frac{(y_1'y_2'-a(x)y_1y_2)'}{y_1'y_2'-a(x)y_1y_2}\right]\cdot Z'-$$
$$-\frac{(y_1'y_2'-a(x)y_1y_2)^2}{4y_1y_1'y_2y_2'}\cdot Z=0$$

The equation (19) is a completely linear homogenous differential equation of the second order, whose coefficients are given with:

(20) $$\begin{cases} B(x)=\dfrac{-(y_1'y_2'-a(x)y_1y_2)^2\cdot(y_1y_2'+y_1'y_2)+2y_1y_1'y_2y_2'(y_1'y_2'-a(x)y_1y_2)'}{2y_1y_1'y_2y_2'(y_1'y_2'-a(x)y_1y_2)} \\ A(x)=-\dfrac{(y_1'y_2'-a(x)y_1y_2)^2}{4y_1y_1'y_2y_2'} \end{cases}$$

In the Ph thesis we have proved that the equation (18) in concrete case (19), if the coefficients *A(x)* and *B(x)* are uninterruptible, determines trigonometry, either elliptic or hyperbolic depending on the sigh of discriminant:

$$D(x)=A(x)-\frac{1}{2}B'(x)-\frac{1}{4}B^2(x)$$

Further, we know, that by transformation

$$Z=e^{-\frac{1}{2}\int B(x)dx}\cdot V(x)$$

where *V=V(x)* is new unknown function, the equation (19) is transformed into canonic form:

(21) $$V''(x)+\left[A(x)-\frac{1}{2}B'(x)-\frac{1}{4}B^2(x)\right]\cdot W(x)=0\,.$$

Thus, depending on sign of the coefficient of the equation (21), there are elliptic, hyperbolic or parabolic type of trigonometry determined by equation (19). Actually, it depends on coefficient *a(x)* of the Hill's equation (1) and more detailed analysis would differentiate these cases. However, this is not our end goal, but the following statement:

Theorem 1. *The solutions of the equation (19) can be oscillating and also set of functions (12)* $Tg*(\cos x*,\sin x*)$ *is trigonometry.*

Definition. Trigonometry *Tg** is called conjugated trigonometry in respect with the trigonometry $Tg(\cos_{a(x)} x,\sin_{a(x)} x$.

Consequence. For harmonic oscillations

$$y''+\lambda^2\cdot y=0$$

where $y_1=\cos\lambda x$ and $y_2=\sin\lambda x$, there are

$$\cos x^* = \sqrt{\cos\lambda x \cdot \lambda\cos\lambda x} = \sqrt{\lambda}\cdot\cos\lambda x,$$
$$\sin x^* = \sqrt{-(-\lambda\sin\lambda x)\cdot\sin\lambda x} = \sqrt{\lambda}\cdot\sin\lambda x,$$

i.e.

$$x^*(x) = \arccos(\sqrt{\lambda}\cdot\cos\lambda x)$$

However, for the coefficients of the equation after short calculation, we obtain

$$A(x) = -\frac{(y_1'y_2' - a(x)y_1y_2)^2}{4y_1y_1'y_2y_2'} = \lambda^2$$
$$B(x) = 0$$

So the equation of the new trigonometry $Tg^*(\cos x^*, \sin x^*)$ is:

$$Z'' + 0\cdot Z' + \lambda^2\cdot Z = 0$$

which is the same as trigonometry of harmonic oscillations. There follows:

Theorem 2. *Harmonic oscillation form a trigonometry which is auto-conjugated with itself*.
For all other cases for which:

$A = A(a(x))$ and $B(a(x)) \neq 0$

$Tg^*(Z_1, Z_2)$ is trigonometry conjugated with trigonometry $Tg(y_1, y_2) = Tg(\cos_{a(x)} x, \sin_{a(x)} x)$.
Based on aforesaid, w conclude that the Hill's equation (1) in which periodic coefficient *a(x)* is positive on canonic interval, has oscillating solutions $y_1 = \cos_{a(x)} x$ and $y_2 = \sin_{a(x)} x$ in the form of iteration sequences which make trigonometry $Tg(\cos_{a(x)} x, \sin_{a(x)} x)$. Also, there is standardized trigonometry $Tg^*(\cos x^*, \sin x^*)$, mutually conjugated with the trigonometry of the Hill's equation. If in the Hill's equation $a(x) = \lambda = const.$, then there are harmonic oscillations which form trigonometry which is auto-conjugated with itself.

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