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OSCILLATING SOLUTIONS OF THE HOMOGENOUS LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER

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Abstract: The classic Sturm's theorems are qualitative theorems that determine a certain number of differential equations which have oscillating solutions through their whole area. Without using Sturm's methods, but using the method of successive approximations, in our work we have solved the canonical equation $y'' + a(x) \cdot y = 0$, in which the coefficient a(x) is a positive function which satisfies Lipschitz's condition. Then, we have directly proved that its solutions are oscillating on canonical interval. In order to have infinite number of zeros, i.e. infinite number of oscillations or infinite asymptotic oscillatority, then the coefficient a(x) which represents difference between forces which cause oscillations and resistance, must be sufficiently big, so that the integral $\int_0^{+\infty} a(x)dx$ could diverge. We have also shown that, if the equation has one oscillating solutions, then all its solutions are oscillating and its zeros are simultaneously inflections (simple non-contacts) of the oscillations. In this case derivatives of the oscillations are some new oscillating functions.

Key words: Differential equations, successive approximations, oscillating solutions, zero oscillations.

It was Sturm who noticed long time ago that solving linear homogenous differential equation of the second order by applying analytical method and sequences provided only local and approximate notion of the solutions. However, it cannot provide the answer to the qualitative question of the global behavior, such as zeros, asymptotics, and stability. That is why he claimed that it was necessary to study directly each individual equation separately. This principle has remained until today, such as equation in electronics by Liénard and Reilly.

However, Sturm could not say which method would make better results possible, because successive approximation and iterations were not known at the time. Without knowing this, we have tried to find it as precisely as possible, by applying iterations, since iterations have more wider range of convergence than sequences, but they require each step to be performed completely and directly during the procedure within one elementary form. Our idea was move within basic theorems on existence of solution, which probably cannot be improved and outmatched. This would lead to us to the conclusion about the number and schedule of zeros, limitation of solutions, asymptotic behavior, periods, rise and stability of solutions. However, we firstly expected to determine conditions which must be fulfilled so that the solutions could be oscillating. In that case we have started from the canonical homogenous linear equation of the second order:

(1) $y'' + a(x) \cdot y = 0$,

in which the coefficient a(x) fulfills the following conditions:

1. a(x)>0 on interval $[0, X]$
2. $a(x) \cdot y$ is an uninterruptible function in the rectangle $D: \begin{cases} 0 \le x \le X \\ a \le y \le b \end{cases}$ in which Lipschitz's condition is also fulfilled.

Then, according to the Picard–Lindelöf theorem the equation (1) has a unique, twice uninterruptible, differential solution y(x) in D area, which can be found by iterations.

The same is determined from the normal form of the equation (1)

(1) $y'' = -a(x)\cdot y$

if we find the first integrals.

(2) $y' = C_1 - \int_0^x a(x)\cdot y(x)dx$ and further

(3) $y = C_1 x + C_2 - \int_0^x\int_0^x a(x)\cdot y(x)dx^2$

If we use relation (3) for defining iteration sequence

$$y^{[1]}(x),\ \ y^{[2]}(x),\dots\ \ y^{[n]}(x),\dots,$$

by applying naturally suggested relation

(4) $y^{[n]}(x) = C_1 x + C_2 - \int_0^x\int_0^x a(x)\cdot y^{[n-1]}(x)dx^2$, n=1,2,3,...

with the initial approximation $y^{[0]}(x)$, which is determined with

$$y^{[0]}(0) = y(0) = y_0,$$

we obtain:

$$y^{[1]}(x) = C_1 x + C_2 - \int_0^x\int_0^x a(x)y^{[0]}x\cdot dx^2$$

$$y^{[2]}(x) = C_1 x + C_2 - \int_0^x\int_0^x a(x)y^{[1]}x\cdot dx^2 =$$

$$= C_1 x + C_2 - \int_0^x\int_0^x a(x)\left[C_1 x + C_2 - \int_0^x\int_0^x a(x)y^{[0]}x\cdot dx^2\right]\cdot dx^2$$

$$y^{[3]}(x) = C_1 x + C_2 - \int_0^x\int_0^x a(x)y^{[2]}x\cdot dx^2 =$$

$$= C_1 x + C_2 - \int_0^x\int_0^x a(x)\left\{C_1 x + C_2 - \int_0^x\int_0^x a(x)\left[C_1 x + C_2 - \int_0^x\int_0^x a(x)y^{[0]}x\cdot dx^2\right]dx^2\right\}\cdot dx^2 =$$

$$= C_1\left\{x - \int_0^x\int_0^x xa(x)dx^2 + \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x xa(x)dx^2 - \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x xa(x)y^{[0]}(x)dx^2\right\} +$$

$$+ C_2\left\{1 - \int_0^x\int_0^x a(x)dx^2 + \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2 - \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)y^{[0]}(x)dx^2\right\}$$

⋮

By mathematic induction, we obtain solution of the equation (1) in the form of iteration sequences.

(5) $y = C_1\left\{x - \int_0^x\int_0^x xa(x)dx^2 + \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x xa(x)dx^2 - \dots\right\} +$

$$+C_2\left\{1-\int_0^x\int_0^x a(x)dx^2+\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2-\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2+..\right\}=$$

$$=C_1\left\{x+\sum_{k=1}^{\infty}(-1)^k\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2...\int_0^x\int_0^x xa(x)dx^2\right\}+$$

$$+C_2\cdot\sum_{k=0}^{\infty}(-1)^k\int_0^x\int_0^x a(x)dx^2...\int_0^x\int_0^x a(x)dx^2$$

=

The integral operator (3) is contracting, i.e. the iteration sequence is convergent based on Picard's theorem.
Since the solution (5) contains two random constants, then (5) is general solution. If we make the choice of constants

(6) $(C_1,C_2)=\begin{matrix}\rightarrow(1,0)\Rightarrow y_2\\ \rightarrow(0,1)\Rightarrow y_1\end{matrix}$

we obtain two particular integrals

(7) $$\begin{cases} y_1=1-\int_0^x\int_0^x a(x)dx^2+\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2-\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2+... \\ y_2=x-\int_0^x\int_0^x xa(x)dx^2+\int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x xa(x)dx^2-..., \end{cases}$$

which are linearly independent, because their Wronskian $W(y_1,y_2)=W(x)$ is different from zero. However, the iterations do not show that solutions (7) are oscillating. The oscillation of the solutions will be proven indirectly, by using condition that the coefficient a(x) is positive.
First, we are going to consider the solution y_1 and form (3):

(8) $$y_1=1-\int_0^x\int_0^x a(x)y_1(x)dx^2$$

It is obvious that for x=0, it is $y_1(0)=1>0$, so it is either $y_1>0$ or $y_1<0$. If $y_1>0$, then, due to $a(x)>0$ and $a(x)\cdot y_1(x)>0$, the integral of the function $a(x)y_1(x)$ is also positive, i.e.

$$\int_0^x a(x)\cdot y_1(x)dx>0,$$

because the integral of the positive function is positive.
In that case double integral is also positive

$$\int_0^x\int_0^x a(x)\cdot y_1(x)dx^2>0,$$

because it is an integral sum whose addends are positive. Then, no matter how randomly small function is

$$a(x)>\varepsilon>0,\ \ y_1(x)>g>0,$$

the following is valid for the double integral

$$\int_0^x\int_0^x a(x)\cdot y_1(x)dx^2 > \varepsilon g\int_0^x\int_0^x dx^2 = \frac{\varepsilon\cdot g}{2!}\cdot x^2$$

Its lower limit can be higher than one, if $\frac{\varepsilon\cdot g}{2!}\cdot x^2 > 1$, i.e. if $x > \sqrt{\frac{2}{\varepsilon\cdot g}}$, so the difference in (8) will be negative, i.e.

$$y_1 = 1 - \int_0^x\int_0^x a(x)y_1(x)dx^2 < 0$$

because double integral does not reach one for sufficiently big $x > \sqrt{\frac{2}{\varepsilon\cdot g}}$. Therefore, the solution y_1, which is monotonously falling and convex on one part of the interval $[0, X]$, is also negative. However, since according to the Picard's theorem on existence of solutions it follows that y_1 is also uninterruptible, there must exist point $\xi\in(0, X)$ which is zero of the integral y_1, i.e. $y_1(\xi) = 0$ which means

$y_1(\xi) = 1 - \int_0^\xi\int_0^x a(x)y_1(x)dx^2 = 0$, i.e. the equation

$$1 - \int_0^x\int_0^x a(x)dx^2 + \int_0^x\int_0^x a(x)dx^2\int_0^x\int_0^x a(x)dx^2 - ... = 0$$

surely has solutions on definite interval $[0, X]$ for sufficiently big X.

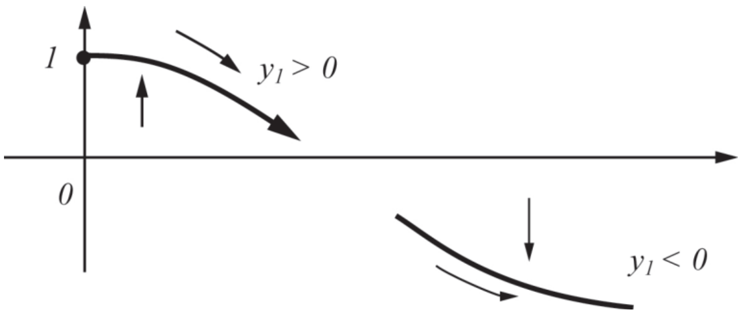

Figure 1.

Completely analogically we prove that if solution y_1 is negative, it cannot be constantly negative, but it is also positive on some sub-interval $[0, X]$. Since it is also uninterruptible, it has at least one zero on definite interval $[0, X]$. Therefore, the solution y_1 is oscillating on definite interval.

With complete analogy we prove that solution y_2, which has the following form

$$y_2 = x - \int_0^x\int_0^x a(x)y_2(x)dx^2$$

is also oscillating. First, for x=0 there is $y_2(0)=0$, so x=0 is already one zero of the function $y_2(x)$. Are there any other zeros? Let $y_2(x)>0$ be permanently on interval $[0, X]$. Since a(x) and $y_2(x)$ are positive functions, in a closed interval $[0, X]$, these have their own infimums and supremums, which means that there are constants, $a, A, m, M \in R^+$, so the following is valid:

$a \le a(x) \le A$ and $m \le y_2 \le M$.

Then for the double integral, the following is valid:

$$am\frac{x^2}{2!} \le \int_0^x\int_0^x a(x)y_2(x)dx^2 \le A \cdot M \cdot \frac{x^2}{2!}$$

Based on the previous non-equality, it follows that

$$y_2 = x - \int_0^x\int_0^x a(x)y_2(x)dx^2 < x - am \cdot \frac{x^2}{2!}$$

However, the square trinomial on the right side, apart from zero x=0, also has zero for $x = \frac{2}{a \cdot m}$. Between these zeros it is positive, while for $x > \frac{2}{a \cdot m}$, it is negative. Therefore y_1 can be smaller than some negative number if $x > \frac{2}{a \cdot m}$, which contradicts the premise that the integral y_2 is constantly positive. This leads to the conclusion that premise is not correct, i.e. y_2 has the second zero ξ apart from x=0 for which $0 < \xi < \frac{2}{a \cdot m}$. Therefore, the solution y_2 on canonical interval $[0, X]$ changes its sign and has at least one more zero due to its uninterruptible nature. Completely analogically we prove that y_1 cannot be constantly negative on definite interval, but it is actually positive on one sub-interval of this interval, which means that it changes the sign, so owing to its uninterruptible nature, it has zeros effectively. This leads to the conclusion that y_2 is oscillating on the interval $[0, X]$.

Theorem 1. The canonical equation (1) for a(x)>0 has oscillating solutions on definite interval $[0, X]$ for sufficiently big X.

In order to preserve oscillations when $x \to +\infty$, i.e. in order to obtain infinite number of oscillations, which also means infinite number of zeros or in order to have infinite asymptotic oscilatority, some additional conditions are necessary for the sequence of the coefficient a(x) size. Therefore, in order to preserve asymptotic stability of solutions, we are going to provide one more theorem.

Theorem 2. If in the equation (1) the coefficient a(x) fulfills following conditions:

1. $a(x) > 0$
2. $\int_0^x a(x)dx$ diverges

then on the interval $[0, +\infty]$ the solutions of the equation (1) are oscillating and have infinite number of zeros.

The condition number 2. requires the positive coefficient a(x) to be sufficiently big in order to cause oscillations. This is in accordance with the physical meaning of the coefficient a(x) which represents difference between the forces which causes oscillations and resistance.

Based on theorems 1 and 2, we conclude that the equation (1), under certain conditions, has oscillating solutions. If it has one oscillating solution, it also has infinite number of them, since $C \cdot y(x)$, where C is a random constant, is oscillating solution if such solution is y(x). The following theorem is valid for the oscillating solutions of the equation (1):

Theorem 3. If canonical equation (1) has one oscillating solution, then all its solutions are oscillating.

<u>Proof.</u>

Let y_1 be oscillating solution, while y_2 is any other solution of the equation (1). Since for solutions of the homogenous, linear differential equation of the second order y_1 and y_2

$$y'' + a(x)\cdot y' + b(x)\cdot y = 0,$$

Liouville's formula is valid.

$$y_2 = y_1 \cdot \int \frac{e^{-\int a(x)dx}}{y_1^2} dx$$

Therefore, for the integrals of the equation (1), since it does not contain the member with the first derivative, the following equality is valid:

$$y_2 = y_1 \cdot \int \frac{dx}{y_1^2}$$

The solution y_1 is supposedly oscillating, which means that it changes sign. Due its uninterruptible nature it also has zeros effectively. That is why the integral $\int \frac{dx}{y_1^2}$ is uncharacteristic in the intervals which contain zeros of the solution y_1. In order the solution y_2 exists (and it exists based on the theorem on existence and uniqueness of solutions) if the integral $\int \frac{dx}{y_1^2}$ is convergent. It means that the product

$$y_1 \cdot \int \frac{dx}{y_1^2}$$

exists and has zeros which are zeros of the solution y_1, if there are no other solutions. Therefore, the solution y_2 is oscillating and since it is a random solution of the equation (1), we conclude that if given equation has one oscillating solution, then all its solutions are also oscillating, which should have been proven.

From the equation

(1) $\quad y'' = -a(x)\cdot y$

and since a(x)>0, it follows that $y'' = 0$, if and only if $y(x) = 0$, i.e. the zeros which exist according to the theorem 1, are simultaneously the inflection points of the solution. There follows:

Theorem 4. All zero solutions (7) of the equation (1) are also inflection points of the solution. These are simple inflection points with slope tangent.

The inflection points of the solution of the equation (1) are not contacts, i.e. they are not inflection points with horizontal tangent. It follows from the derivative of the equation (1):

(9) $\quad y''' = -a'(x)\cdot y - a(x)\cdot y'(x)$

If ξ is the zero solution of y(x), then $y(\xi) = 0$, $a(\xi) > 0$. So, there is:

$$y'''(\xi) = -a'(\xi)\cdot y(\xi) - a(\xi)\cdot y'(\xi) \neq 0$$

because $y'(\xi) \neq 0$, since according to one classical theorem, solutions of the linear differential solution of the second order cannot have double zeros (contacts).
However, from (9), there follows that derivative $a'(x)$ and its sign are important for the quality of both inflections and zero oscillating solutions of the equation (1), i.e. it is important whether $a(x)$ is monotonic or not. However, Sturm could not realize this due to his algebraic approach to the problem.
Theorem 4 means that, if $a(x)$ is monotonic and positive function, the solutions of the equation (1) given with (7) between every two successive zeros are either only concave or only convex, i.e. it is either $y'' > 0$ or $y'' < 0$. However, if ξ_n and ξ_{n+1} are two successive zero solutions of, let us say y_1, then due to its uninterruptible and differentiable nature and based on Rolle's theorem, there is at least one zero of the derivative $y_1'(x)$. This leads to the following

Theorem 5. Between every two successive zeros of the solution (7), there is at least one zero of their derivative, i.e. if
$y_1(\xi_n) = 0$, $y_1(\xi_{n+1}) = 0$

then, there is $\eta_n, \zeta_n < \eta_n < \xi_{n+1}$, where $y_1'(\eta_n) = 0$
The same is valid for the solution y_2..
The Rolle's theorem allows for several zeros of the derivative between two zero functions in case of its multiple non-monotony.

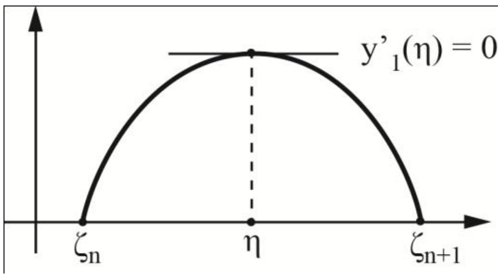

Figure 2.

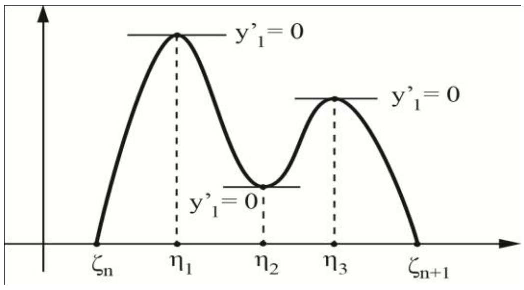

Figure 3.

The right side of the equation (1) $y'' = -a(x) \cdot y = F(x,y)$ behaves completely differently if coefficient a(x) is not monotonic function, because then F(x, y) as non-monotonic function has multivalued inverse function which naturally influences both oscillatority of the solution and zeros of the oscillations. That is why, if a(x) is not a monotonic function, first all intervals of the monotony of the coefficient a(x) should be determined. Then, oscillating solutions, number of zeros and locations of zeros should be determined in each of those intervals. Sturm sensed this to some part when he formulated his famous theorem on minimum number of zero oscillations when coefficient a(x) is strictly monotonic function on definite interval $[0, X]$. However, he did not have better mathematical apparatus in order to prove this.
Functional sequencces which define oscillating solutions (7) of the equation (1) are uniformly convergent, because they have been majorated with convergent potential sequences. That is why they can be differentiated member by member and their first derivatives are given with:

(10) $y_1' = -\int_0^x a(x)y_1(x)dx$

(11) $y_2' = 1 - \int_0^x a(x)y_2(x)dx$

From the formula (11), by applying partial integration:

$U = y_2$, dv=a(x)dx

$dU = y_2'(x)dx$, $v = \int a(x)dx$
we obtain

(12) $y_2' = 1 - y_2 \int a(x)dx + \int \left(\int a(x)dx\right) y_2'(x)dx$
So in the zero ξ_1 of the function $y_2(x)$ there is

(13) $y_2'(\xi_1) = 1 + \int_0^{\xi_1} \left(\int a(x)dx\right) \cdot y_2'(x)dx$

So there follows:

Theorem 6. Tangents (slopes) of the solution $y_2(x)$ are given with the formula (12), while tangents in the zeros of the solutions are given with the formula (13).
There is similar for $y_1'(x)$ from (11)

$U = y_1(x)$, dv=a(x)dx

$dU = y_1'(x)dx$, $v = \int a(x)dx$, so there is
$y_1'(x) = -y_1(x)\int a(x)dx + \int \left(\int a(x)dx\right) y_1'(x)dx$
In the zero ξ_2 of the function $y_1(x)$, there is $y_1(\xi_2) = 0$. So we obtain:

(14) $y_1'(\xi_2) = \int_0^{\xi_2} \left(\int a(x)dx\right) \cdot y_1'(x)dx$

Based on the previous, we formulate an important theorem:

Theorem 7. Derivatives (10) and (11) of the oscillating solutions of the equation (1) are also some new oscillating functions.
Sturm's theorems directly refer to the relations between location of zeros of the oscillating solutions y_1 and y_2 and their derivatives y_1' and y_2'.

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