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A STOCHASTIC MODEL FOR OPTIMAL INVESTMENTS

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Abstract: The paper presents a stochastic model of an investor's portfolio optimization. The optimal control policies are based on stochastic dynamic programming. It continues the research of the mathematical dynamical model applied to the study of a financial market in discrete time presented in [6], where the model was used for the technical evaluation of some options of European type. The results in this paper synthesize, from a different approach, some results obtained in [3] and in [4] for the optimization problem. The mathematical framework, which emphasizes the stochastic dynamic programming equation for a control problem, is constructed on results given in [1], [2] and [5]. The aim of this paper is to establish the dynamics of the portfolio and its optimal level, by describing a stochastic model for optimal investments. A particular case of a portfolio consisting of two assets, one risky and the other without risk, is also presented.

1. INTRODUCTION

The general mathematical model takes into consideration a number of *n* financial instruments. We consider that their prices are influenced by several economic factors. As a particular case, we will consider the classic Merton type problem of the portfolio optimization which assumes that an investor assigns dynamically his wealth for two assets, one of them risky and another without any risk and chooses an optimal consumption rate in order to maximize the global expected utility of consumption in a finite, respectively infinite horizon.

2. THE DYNAMIC PROGRAMMING EQUATION FOR A STOCHASTIC CONTROL PROBLEM

Let us consider that, in a continuous time model, uncertainty in economy is described through a probability space denoted by (Ω, F, P), where F is the family of all subsets of the finite nonempty set Ω, and P denotes the probability measure such that $P(\{\omega\}) > 0$ for all $\omega \in \Omega$. Let us consider the filter $(F_t)_{t\geq 0}$, which is a σ-algebra that that captures the basic information available at the moment *t.*

In what follows, we will define a dynamic programming equation for a stochastic control problem over a finite time interval $[0,T]$. Let us denote $\Sigma \subset R^n$ the state space and $U \subset R^n$ the control space.

Let us consider that the evolution of a controlled process $X_t = X_t(\theta_t)$ is given by the equation

$$dX_t(\theta_t) = \sigma(t, X_t, \theta_t)dt + \sigma(t, X_t, \theta_t)dW_t + \int_\Gamma q(t, X_t, \rho, \theta_t)N(dt, d\rho) \qquad (2.1)$$

where $(W_t)_{t\geq 0}$ is a *n*-dimensional Brownian motion, N is a stochastic measure on $R \times R^n$, $\theta_t \in U$, the function $\sigma : R \times \Sigma \times U \to R^n$ is continuous and describes the relation, in time, between X_t and θ_t, the function $q : R \times \Sigma \times R^n \times U \to R^n$ is measurable and bounded and gives the jumps depending on the state and on the control, and $\Gamma \subset R^n$ is a compact set.

Definition 2.1. An adapted control process θ_t with values in the control space U, which satisfies the condition

$$E\left[\int_0^T |\theta_s|^m ds\right] < \infty, m \geq 1, \tag{2.2}$$

Is called *admissible strategy.* The set of all admissible strategies is denoted by Θ.

Remark 2.2. If the control space U is compact, then we have $|\theta_s| \leq M$ for some $M < \infty$, and, hence, relation (2.2) is verified.

In what follows, let us consider the continuous and bounded applications $k : [0,T] \times \Sigma \times U \to R$, representing the current cost, and $g : \Sigma \to R$, representing the final cost. The expected cost of the controlled state process $X_t(\theta_t)$, where $X_t = x \in \Sigma$ and $\theta_t \in \Theta$ is an admissible strategy, is given by the relation

$$W(t,x,\theta_t) = E_{tx}\left[\int_t^T k(s, X_s, \theta_s)ds + g(X_T)\right]. \tag{2.3}$$

Let $V \in C^{1,2}\left([0,T] \times R^n\right)$ be a classic solution of the Hamilton-Jacobi-Bellman equation

$$\begin{gathered} \inf_{\theta \in U}\left[A^\theta V(t,x) + k(t,x,\theta)\right] = 0, t < T \\ V(T,x) = g(x) \end{gathered} \tag{2.4}$$

where A^θ denotes the generator of the state process $X_t(\theta_t)$. As the state process is a solution of the equation (2.1), we obtain that A^θ is given by the relation

$$A^\theta h(t,x) = h_t(t,x) + b h_x(t,x) + \frac{1}{2} Tr\left[\sigma(t,x,\theta_t)\sigma(t,x,\theta_t)^* h_{xx}(t,x)\right] +$$
$$+ \lambda \int_\Gamma \left[h(t, x + q(t,x,z,\theta_t)) - h(t,x) - q(t,x,z,\theta_t) h_x(t,x)\right] dz,$$

where $h \in C^{1,2}\left([0,T] \times \Sigma\right) \cap D(A)$. Let us suppose that V and its partial derivatives V_t, V_x and V_{xx} satisfy a polynomial growth condition.

Theorem 2.3. *Following relations hold:*

(i) $V(t,x) \leq W(t,x,\theta_t),\ \forall \theta_t \in \Theta$;

(ii) If, for $(t,x) \in [0,T] \times \Sigma$, *there exists* $\theta_t^* \in \Theta$ *such that*

$$\theta_t^* \in \arg\min\left[A^{\theta_s^*} V(s, X_s^*) + k(s, X_s^*, \theta_s^*)\right], \quad \forall (s,\omega) \in [0,T] \times \Omega$$

then $V(t,x) = W(t,x,\theta_t^*)$.

Proof. (i) As V is a classic solution of the Hamilton-Jacobi-Bellman equation (2.4), following relation holds for all admissible strategy $\theta_t \in \Theta$

$$A^{\theta_s} V(t,x) + k(t,x,\theta_s) \geq 0, \forall s \in [0,T].$$

We also have

$$E_{t,x}\left[h(s, X_s)\right] - h(t,x) = E_{t,x}\left[\int_t^s A^{\theta_r} h(r, X_r) dr\right], t < s, s \in [0,T]$$

for all Markov processes X_t and bounded $h \in D(A)$. It follows that

$$E_{t,x}\left[V(s, X_s)\right] - V(t,x) = E_{t,x}\left[\int_t^s A^{\theta_r} V(r, X_r) dr\right], t < s, s \in [0,T]$$

and, further

$$V(t,x)=E_{t,x}\left[-\int_t^s A^{\theta_r}V(r,X_r)dr+V(s,X_s)\right], t<s, s\in[0,T].$$

For $s = T$ and according to relation (2.3) we obtain

$$V(t,x)\le E_{t,x}\left[\int_t^s k(r,X_r,\theta_r)dr+V(s,X_s)\right]=W(t,x,\theta_t).$$

(ii) As following equality holds for all $(t,\omega)\in[0,T]\times\Omega$

$$-A^{\theta_t^*}V(t,X_t^*)=k(t,X_t^*,\theta_t^*),$$

we obtain that $V(t,x)=W(t,x,\theta_t^*)$, which ends the proof. ☐

3. STOCHASTIC MODEL FOR OPTIMAL INVESTMENTS

We consider a portfolio consisting of n assets. The actual value for the asset i, where $i=\overline{1,n}$, is denoted by P_i and the investments return by R_i, over a time horizon T. The returns are normal distributed with mean $\mu_i T$ and quadratic variance $\sigma_i\sqrt{T}$. The correlation coefficient between the returns of the assets i and j is denoted by ρ_{ij}.

Let us suppose that the dynamics of the price of the asset i is given by the evolution equation, which represents a diffusion process

$$\frac{dP_i}{P_i}=\mu_i(z(t))+\sum_{j=\overline{1,m}}\rho_{ij}(z(t))dW_j(t), t\in[0,T], i=\overline{1,n}, \quad (3.1)$$

where the vector $W(t)=(W_1(t),W_2(t),...,W_m(t))$ is a m-dimensional Brownian motion which emphasizes the uncertainty of some elements of the stochastic model and the vector $z(t)=(z_1(t),z_2(t),...,z_l(t))$ synthesizes the economic factors that influence the prices of the assets and has the ergodicity property. The dynamics of those factors is described by the evolution equation

$$dz_k(t)=h_k(z(t))dt+\sum_{j=\overline{1,m}}\sigma_{kj}(z(t))dW_i(t), k=\overline{1,l}. \quad (3.2)$$

The parameter μ is called *drift rate*, *expected return* or *growth rate* of the asset and σ is the *volatility* of the asset. These parameters have different effects on the price evolution. The drift is not evident in a short period of time, while the volatility is dominant.

Let us denote by $Y(t)$ the investor's capital, with initial value equal 1, and $y_i(t)$ the proportion attributed to the asset i, $i=\overline{1,n}$. The dynamics of the capital is given by the equation

$$\frac{dY}{Y}=r(t)+\sum_{i=\overline{1,n}}y_i(t)[\mu_i(z(t))-r(t)]dt+\sum_{i=\overline{1,n}}\sum_{j=\overline{1,m}}y_i(t)\rho_{ij}(z(t))dW(t). \quad (3.3)$$

The solution of equation (3.3) is, according to Theorem 2.3, given by

$$Y(t)=E\left[\int_0^T\left\{r(t)+\sum_{i=\overline{1,n}}y_i(t)[\mu_i(z(t))-r(t)]dt\right\}\right]+$$
$$+E\left[\int_0^T\sum_{i=\overline{1,n}}\sum_{j=\overline{1,m}}y_i(t)\rho_{ij}(z(t))dW(t)-\frac{1}{2}\int_0^T\sum_{j=\overline{1,m}}\left[\sum_{i=\overline{1,n}}y_i(t)\rho_{ij}z(t)\right]^2dt\right].$$

In infinite horizon, as we consider $t\to\infty$, then $V(t,z)$ will be approximated by the relation *V(x)+tM*, where

$$M=\frac{1}{2}\sum_{i=1,n}\sum_{j=1,m}a_{ij}(z)D_{ij}V(z)+\frac{1}{2}\sum_{i=1,n}\sum_{j=1,m}a_{ij}(z)D_iV(z)D_jV(z)+$$
$$+\left\{h(z)+\frac{\gamma}{1-\gamma}\sigma(z)\varsigma(z)a(z)\left[\mu(\bar{z}(t))-r(t)\right]\right\}V_{zz}(z)+$$
$$+\frac{1}{2}\frac{\gamma}{1-\gamma}a(z)\left[\mu(\bar{z}(t))-r(t)\right]+\gamma r(t)$$

and $\gamma\in(0,1)$ is a risk sensitive parameter, characteristic for the HARA type utility function.

The optimal value of the portfolio is given by the relation

$$u(z)=\frac{1}{1-\gamma}a(z)\left[\mu(\bar{z}(t))-r(t)+\sigma^2(z)V_{zz}(z)\right]$$

Particular case. Let us consider a portfolio consisting of a risky asset with the price at some moment *t* denoted by *P(t)* and one non risky asset with a constant interest rate $r > 0$. If we denote *Z(t)* = ln *P(t),* then, according to (3.2), we obtain following relation

$$dZ(t)=\alpha\left[\mu t+Z_0-Z(t)\right]dt+\sigma dW(t),$$

where α denotes a constant depending on the type of the asset. The economic factor that influences the price is considered given by the relation

$$z(t)=Z(t)-Z_0-\mu t+\mu\alpha^{-1}$$

which implies that

$$dz(t)=-\alpha z(t)dt+\sigma dW(t).$$

Hence, according to (3.1), the dynamics of the price of the risky asset is given by

$$\frac{dP(t)}{P(t)}=\left[-\alpha z(t)+\mu+\frac{\sigma^2}{2}\right]dt+\sigma dW(t).$$

The dynamic programming equation for the optimization of the portfolio will be written as

$$V=\frac{\sigma^2}{2}V_{yy}(y)+\frac{\sigma^2}{2}\cdot\frac{1}{1-\gamma}V_y(y)^2+\left[-\alpha\frac{1}{1-\gamma}x+\frac{\gamma}{1-\gamma}\left(\frac{1}{2}\sigma^2+\mu-r\right)\right]V_y(y)+$$
$$+\frac{1}{2\sigma^2}\cdot\frac{\gamma}{1-\gamma}\left(\frac{1}{2}\sigma^2+\mu-r-\alpha x\right)^2+\gamma r.$$

It follows that the optimal portfolio will be given by

$$y(x)=\frac{\alpha}{\sigma^2\sqrt{1-\gamma}}x+\frac{1}{\sigma^2}\left(\frac{1}{2}\sigma^2+\mu-r\right),\ \gamma\in(0,1).$$

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