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THE ANALYSIS OF THE INFLUENCE OF DISTRIBUTED MASS OF THE SPRINGS ON THE RESONANCE OF THE ELASTICAL MECHANICAL SYSTEMS

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Abstract: This article presents the phases and the final results of the elaboration of the physical and mathematical models of 1DOF elastic mechanical systems taking into consideration the distributed mass of two kind of embedded beam spring: axial spring and torsional spring. The result and final considerations have a real utility in fast and operational calculus of the natural frequencies of this kind of mechanical models, pointing out the influence of the distributed mass of the spring beam for resonance characteristic.

1. INTRODUCTION

The 1DOF mechanical systems has minimum one mass element and one axial spring, usually being modeled like in figure no. 1 a). Every system with one freedom degree (1DOF), no matter how many elements (mass elements, springs), neglecting damping elements, can be reduced to the model of the figure 1 a). In the same consideration, the 1DOF mechanical systems with f rotational movement has minimum one inertial element and one elastic twist element like in the figure 2 a).

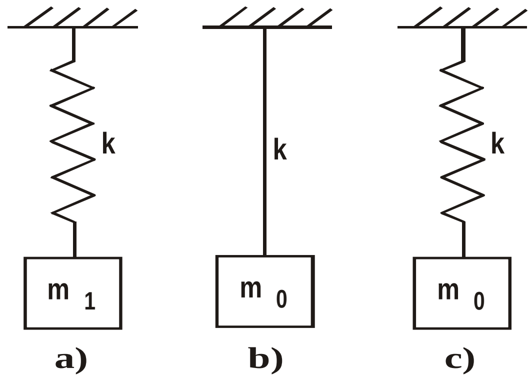

Fig. 1 The calculus models for 1DOF system with distributed mass axial spring
a) the system neglecting the spring mass
b) the system with total equivalent mass
c) the equivalent system with coil spring

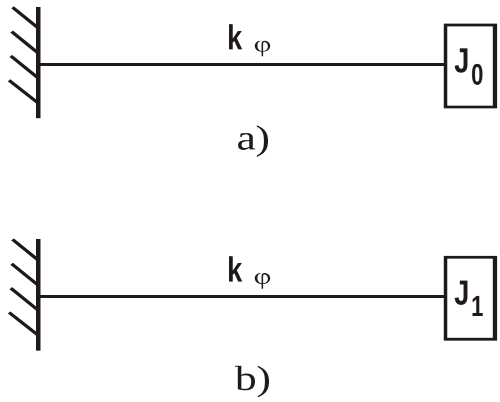

Fig. 2 The calculus models 1DOF system with distributed mass torsional spring
a) the system neglecting the spring mass
b) the system with total equivalent mass

The resonance pulsation for the reduced model with axial spring is

$$p = \sqrt{\frac{k}{m_1}}\ , \tag{1}$$

and the natural frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_1}} \ , \qquad (1a)$$

where m_1 is the reduced mass of the system (neglecting the spring mass) and k is the coefficient of stiffness.

In the similar way, the resonance pulsation for the calculus model of the 1DOF system with torsional spring from figure 2 a) is

$$p_0 = \sqrt{\frac{k_\varphi}{J_0}} \ , \qquad (2)$$

and the natural frequency is

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k_\varphi}{J_0}} \ , \qquad (2a)$$

where J_0 is the reduced inertia of the system (neglecting the torsional beam spring mass) and k_φ is the torsional coefficient of stiffness.

2. PHYSICAL AND MATHEMATICAL MODELS OF 1DOF SYSTEM WITH AXIAL SPRING DISTRIBUTED MASS

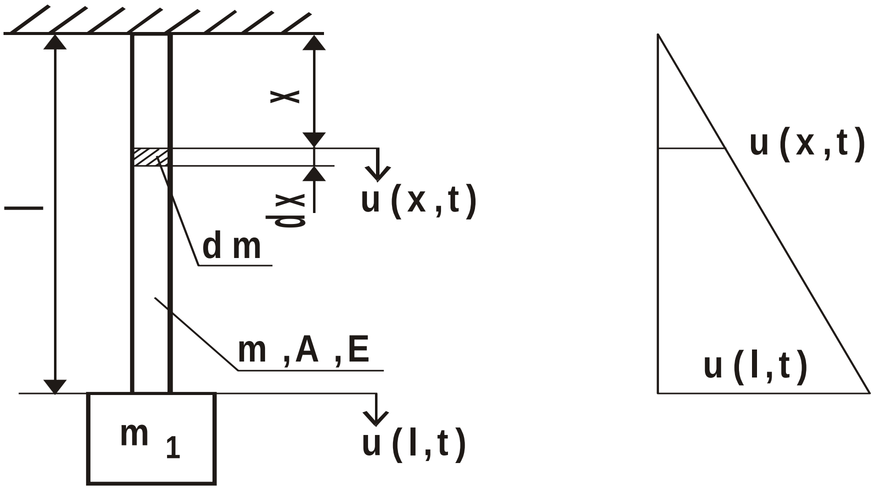

Fig. 3 The calculus diagrams for resonant equivalent mass (1DOF system with distributed mass axial spring)

2.1. Physical model of the system with axial spring

The figure no. 3 shows the calculus diagrams for a simple mechanical system with an axial spring bend and a single concentrated mass in translation movement of vibration. For simplification, it consider an homogeneous material for the beam and a constant transversal section, therefore the value of the produce EA is constant, where E is the elastic modulus and A is the area of the section.

The linear specific mass of the beam is

$$\rho_l = \frac{m}{l} \ , \qquad (3)$$

the mass of the infinitesimal hatching element being

$$dm = \rho_l dx = \frac{m}{l} dx \qquad (4)$$

The study of the reducing mass of the spring can be done in two different hypothesis for the calculus of the static deformation of the beam:

1)taking into consideration only the weight of mass m_1;

2)taking into consideration the total weight of mass m_1 and the mass m of the beam.

2.2. Mathematical model of the system with axial spring

2.2.1.Calculus of the static elongation done by the mass m_1

According to Hooke law, the elastic elongation of the section located on the distance x from the clamping is

$$u(x,t) = \frac{G_1}{EA} x \qquad (5)$$

and the displacement of the beam's end is

$$u(l,t) = \frac{G_1}{EA} l \qquad (6)$$

The relation between the two displacement is

$$u(x,t) = \frac{x}{l} u(l,t) \ , \qquad (7)$$

through derivation obtaining the relation between the velocities:

$$\dot{u}(x,t) = \frac{x}{l} \dot{u}(l,t) \ , \qquad (8)$$

The kinetic energy of the element dm located on the distance x is

$$dE = \frac{1}{2} dm \cdot [\dot{u}(x,t)]^2 = \frac{1}{2} \frac{m}{l} \left(\frac{x}{l}\right)^2 [\dot{u}(l,t)]^2 dx \qquad (9)$$

The energy for the entire beam is obtained by integration

$$E = \int dE = \int_0^l \frac{1}{2} \frac{m}{l} \left(\frac{x}{l}\right)^2 [\dot{u}(l,t)]^2 dx \qquad (10)$$

After the calculus of the integral (10), it results the total kinetic energy of the beam as follows:

$$E = \frac{1}{2} \frac{m}{l^3} [\dot{u}(l,t)]^2 \left. \frac{x^3}{3} \right|_0^l = \frac{1}{2} \frac{m}{3} [\dot{u}(l,t)]^2 \qquad (11)$$

Since the general relation of the kinetic energy of the reduced mass is

$$E = \frac{1}{2} m_r v^2 \ , \qquad (12)$$

where m_r is the reduced mass and v is the velocity of the section where the reducing calculus is to be done, through identification of the inertial terms from (11) and (12) it obtains the expression of the reduced/equivalent mass of the beam on the end of it:

$$m_r \equiv m_{eqv} = \frac{m}{3} \qquad (13)$$

The total kinetic energy of the mechanical system is obtaining by summing the energy of the beam and the energy of the mass element m_1

$$E = E_{beam} + E_{m_1} = \frac{1}{2} m_0 [\dot{u}(l,t)]^2 \ , \tag{14}$$

where m_0 is the total reduced/equivalent mass of the system in the section end of the beam (where is located the mass m_1); the diagram of the equivalent system is shown in figure 1 b).

Through the identification of the terms of the kinetic energies, it obtains expression of the total reduced/equivalent mass:

$$m_0 = \frac{m}{3} + m_1 \tag{15}$$

Analyzing the relation (15), it may take the next preliminary conclusions:
-one third of the beam mass is taking into consideration for the calculus of the total kinetic energy of the system;
-the total mass of the system is bigger than the mass m_1, therefore the value of resonance pulsation is decreasing;
-the equivalent calculus diagram is shown in figure 1 c), the resonance pulsation and the natural frequency having the following expressions:

$$p_{eqv} = \sqrt{\frac{k}{m_1}} \tag{16}$$

$$f_{eqv} = \frac{1}{2\pi}\sqrt{\frac{k}{m_1}} \tag{17}$$

2.2.2.Calculus of the static elongation done by the mass m_1 *and the beam mass* m

Taking into consideration that the mass of the beam is homogeneous distributed, the linear specific weight of it is

$$q = \frac{mg}{l} \ , \tag{18}$$

where q is the specific weight and g is the gravity acceleration.

The total force which produces the displacement of the section situated on the distance x from the clamping is done by summing the weight m_1 and the weight of the part of the beam located under this section. The expression of the total force is

$$F = q(l - x) + G_1 = g\left[m\left(1 - \frac{x}{l}\right) + m_1\right] \tag{19}$$

According to Hooke law, for section x it may write

$$\frac{F}{A} = E\frac{u(x,t)}{x}, \tag{20}$$

from where the displacement of the section is

$$u(x,t) = \frac{F}{EA}x \tag{21}$$

Taking into consideration the expression (19) of the force F, the differential of the displacement done by (21) is

$$du(x,t) = \frac{F}{EA}dx = \frac{g}{EA}\left[(m_1 + m) - m\frac{x}{l}\right]dx \tag{22}$$

Integrating the relation (22), it can obtains the expression of the displacement of any section of the beam. Thus, the displacement of the section x is as follows:

$$u(x,t)=\int_0^x du(x,t)=\frac{g}{EA}\int_0^x\left[(m_1+m)-m\frac{x}{l}\right]dx \qquad (23)$$

Consequently, the displacement of the beam's end has the expression:

$$u(l,t)=\int_0^l du(x,t)=\frac{g}{EA}\int_0^l\left[(m_1+m)-m\frac{x}{l}\right]dx \qquad (24)$$

By calculating the integrals (23) and (24), it obtains the final expressions for the displacements of the section x and of the end of the beam as follows:

$$u(x,t)=\frac{gx}{EA}\left[(m_1+m)-m\frac{x}{2l}\right] \qquad (25)$$

$$u(l,t)=\frac{gl}{EA}\left(m_1+\frac{m}{2}\right) \qquad (26)$$

From (25) and (26) it can write the relation between the two displacements

$$u(x,t)=\lambda u(l,t)\ , \qquad (27)$$

where λ is a constant parameter as follows:

$$\lambda=\frac{x}{l}\frac{2m_1+m\left(2-\frac{x}{l}\right)}{2m_1+m} \qquad (28)$$

Deriving (27), it obtains the relation betweens the velocities of the section x and the end of the beam:

$$\dot{u}(x,t)=\lambda\dot{u}(x,t) \qquad (29)$$

The kinetic energy of the element dm is

$$dE=\frac{1}{2}dm[\dot{u}(x,t)]^2=\frac{1}{2}\frac{m}{l}dx\cdot\lambda^2[\dot{u}(l,t)]^2\ , \qquad (30)$$

or

$$dE=\frac{1}{2}\frac{m}{l}\left[\frac{x}{l}\frac{2m_1+m\left(2-\frac{x}{l}\right)}{2m_1+m}\right]^2[\dot{u}(l,t)]^2dx \qquad (31)$$

The total kinetic energy of the beam is calculating by integrating the infinitesimal energy done by the relation (31)

$$E=\int dE=\frac{1}{2}A\cdot I\cdot[\dot{u}(l,t)]^2\ , \qquad (32)$$

where:

$$A=\frac{m}{l^3(2m_1+m)^2}=\text{const} \qquad (33)$$

$$I=\int_0^l\left[2(m_1+m)x-\frac{m}{l}x^2\right]^2dx \qquad (34)$$

After the calculus, the expression of the integral (34) is

$$I=\frac{l^3}{15}\left(20m_1^2+25m_1m+8m^2\right), \qquad (35)$$

thus the total kinetic energy of the beam is

$$E = \frac{1}{2}\frac{20m_1^2 + 25m_1 m + 8m^2}{15(2m_1 + m)^2} m \cdot [\dot{u}(l,t)]^2 \qquad (36)$$

Because the reduced mass of the beam is to be done in the end of it, the expression of the kinetic energy it have to be as follows:

$$E = \frac{1}{2} m_r [\dot{u}(l,t)]^2 \qquad (37)$$

Through the identification of the terms of the relations (36) and (37), the equivalent/reduced mass of the beam in the end of it has the expression

$$m_r \equiv m_{eqv} = \frac{20\mu^2 + 25\mu + 8}{15(2\mu + 1)^2} m = \beta(\mu) \cdot m \ , \qquad (38)$$

where the dimensionless parameters μ and β are the follows:

$$\mu = \frac{m_1}{m} \qquad (39)$$

$$\beta(\mu) = \frac{20\mu^2 + 25\mu + 8}{15(2\mu + 1)^2} \qquad (40)$$

2.2.3.Partial conclusions for 1DOF system with axial spring

Analyzing the relations (38)-(40), it may take the next partial conclusions:

-only one fraction of the beam mass is taking into consideration for the calculus of the total kinetic energy of the system;

-the fraction of beam mass which participates into the calculus of the equivalent mass depends on the mass m_1; this fraction is described by the parameter β which is bigger if the mass of the end of the beam is smaller and inversely

-the maximum influence of the beam mass is reached in absence of the mass m_1:

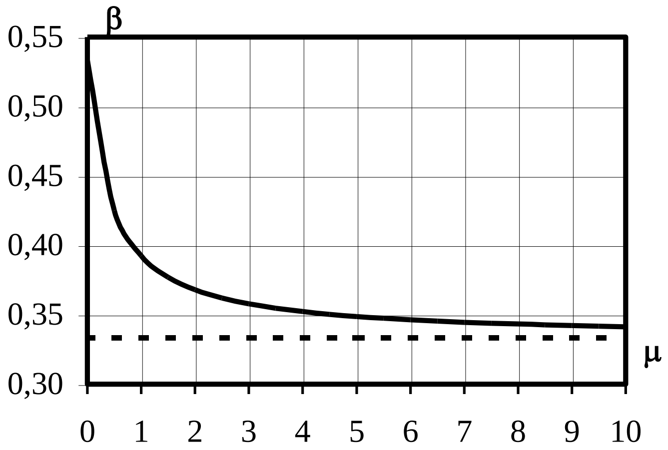

Fig. 4 The variation of the β parameter depending on "dimensionless mass" μ

$$\beta_{max} = \beta(0) = \frac{8}{15} \Rightarrow m_{r\,max} = \frac{8}{15} m \qquad (41)$$

-the influence of the beam mass is decreased if the value of mass m_1 is very big:

$$\beta_{min} = \beta(\infty) = \frac{1}{3} \Rightarrow m_{r\,min} = \frac{1}{3} m \qquad (42)$$

-the variation of the parameter β (which measures influence mass of the beam on the equivalent mass) depending on the "dimensionless mass" μ) is shown in the figure no. 4.

3. PHYSICAL AND MATHEMATICAL MODELS OF 1DOF SYSTEM WITH TORSIONAL SPRING DISTRIBUTED MASS

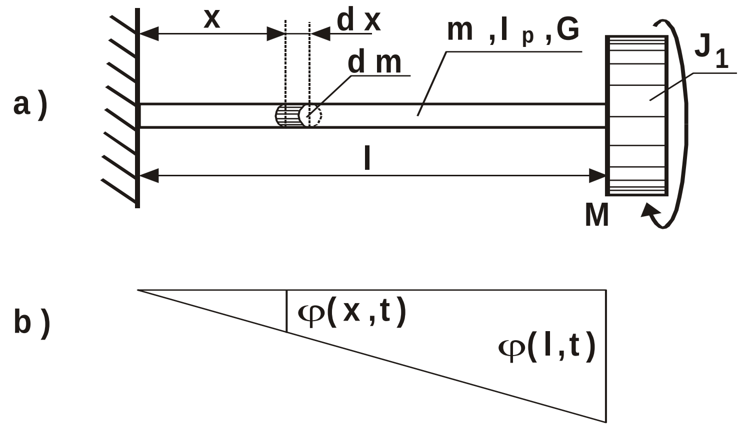

Fig. 5 The calculus diagrams for resonant equivalent mass (1DOF system with distributed mass torsional spring)

3.1. Physical model of the system with torsional spring

The figure no. 5 shows the calculus diagrams for a simple mechanical system with rotational movement and a torsional bend spring with circular section.

It presumes that all the dimensional, elastic and inertial characteristics of the system are known:

●for the axial mounted flywheel: the mass and the axial inertia J_0

●for the torsional beam: the mass and the axial inertia J, the length l, the polar inertia I_p, the torsional coefficient of stiffness

Without to make a particular approach of this problem, it considers that the fixed beam is made from homogeneous material and the sectional area is a constant circular shape.

3.2. Mathematical model of the system with torsional spring

The figure 5 shows the diagram for calculus of the equivalent inertia of the system flywheel-torsional beam.

According to Hooke law, if G is the modulus of torsion, on the acting of a torsional moment M_t, the rotation displacements φ are:

▪for the section x

$$\varphi(x,t) = \frac{M}{GI_p}x \tag{43}$$

▪for the end of the beam

$$\varphi(l,t) = \frac{M}{GI_p}l \tag{44}$$

From (43) and (44), the relation between the two rotation displacements is:

$$\varphi(x,t) = \frac{x}{l}\varphi(l,t) \tag{45}$$

Deriving the relation (5), it obtains the relation between the spin velocities as follows:

$$\dot{\varphi}(x,t) = \frac{x}{l}\dot{\varphi}(l,t) \tag{46}$$

Presuming that the section of the beam is full circular where r is the radius of it, the hatchet element dm from the section x has the inertia

$$dJ = \frac{dm}{2}r^2 = \frac{1}{2}\frac{m}{l}dx \cdot r^2 \tag{47}$$

If the torsional beam is a pipe with $2R$ outside diameter and $2r$ inside diameter, the inertia of the infinitesimal mass dm is

$$dJ = \frac{1}{2}\frac{m}{l}dx\left(R^2 - r^2\right) \tag{48}$$

and the kinetic energy is as follows:

$$dE = \frac{1}{2}dJ[\dot{\varphi}(x,t)]^2 \tag{49}$$

Taking into consideration the relations (46) and (47), the kinetic energy becomes:

$$dE = \frac{1}{2}\frac{mr^2}{2l^3}[\dot{\varphi}(l,t)]^2 x^2 dx \tag{50}$$

The total kinetic energy of the beam is to get by integrating the relation (10), the final expression of it being

$$E = \int dE = \frac{1}{2}\frac{J}{3}[\dot{\varphi}(l,t)]^2 \ , \tag{51}$$

where $J = \frac{1}{2}mr^2$ is the inertia of the entire torsional beam (full section).

Because the kinetic energy of the beam calculated with the rotational velocity of the end section has the expression

$$E = \frac{1}{2}J_r[\dot{\varphi}(l,t)]^2 \ , \tag{52}$$

the equivalent inertia of the beam in this section is obtained from (51) and (52) as follows:

$$J_{eqv} = \frac{1}{6}mr^2 = \frac{J}{3} \tag{53}$$

Consequently, the calculus model is that shown in the figure 2 b) and the total equivalent inertia of the system is

$$J_1 = J_{eqv} + J_0 = \frac{J}{3} + J_0 \tag{54}$$

4. CONCLUSIONS

The physical and the calculus model of the mechanical system with an axial elastic beam and a concentrated mass on the end of it make evident the influence of the distributed mass of the spring on the resonance behavior. This influence is bigger how much more the beam mass is bigger and the end mass is smaller.

In any case, the total mass of the system is bigger than the concentrated mass m_1, therefore the value of resonance pulsation is decreasing. The modification of the calculus mass and of the resonance frequency is very important especially for the machines with the resonant function regime.

The physical and the calculus model of the mechanical system with a torsion elastic beam and a flywheel on the end of it make evident the influence of the distributed mass of the beam on the resonance behavior. This influence is bigger how much more the beam mass is bigger and the end mass is smaller. In any case, the equivalent inertia mass of the system is bigger than the inertia of the flywheel, therefore the value of resonance pulsation is decreasing. The modifications of the calculus inertia and of the resonance frequency are very important because can lead to major modifications of the dynamic behavior of the system, especially for transient regimes (starting, stopping).

References:

[1] Bratu, P. - „Mechanical elastic systems vibration (Vibratiile sistemelor elastice)", Editura Tehnica, Bucharest, 2000
[2] Buzdugan, Gh., Fetcu, L, Rades, M. - „Mechanical vibration (Vibratii mecanice)", Editura Didactica si Pedagogica, Bucharest, 1982
[3] Dragan, N. - „Machines dynamics (Dinamica masinilor)", "Dunarea de Jos" University of Galati, Galati, 2005
[4] Radoi, M., Deciu, E.. - „Mechanics (Mecanica)", Editura Didactica si Pedagogica, Bucharest, 1977