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CONSIDERATIONS ON THE INFLUENCE OF DISTRIBUTED MASS OF THE BENDING SPRINGS ON THE RESONANCE OF THE 1DOF ELASTICAL MECHANICAL SYSTEMS

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Abstract: This study presents the phases and the final result of the elaboration of the physical and mathematical models of 1DOF elastic mechanical systems with bending springs, taking into consideration the distributed mass of the horizontal clamping beam. The result and final considerations have a real utility in fast and operational calculus of the natural frequencies of this kind of mechanical models, pointing out the influence of the mass of bending spring beam for resonance characteristics.

1. INTRODUCTION

The mechanical systems with one freedom degree has minimum one mass element and one elastic element, usually being modeled like in *figure 1 a)*. Every system with one freedom degree, no matter how many elements (mass elements, elastic/spring elements, neglecting damping elements) can be reduced to the model of the *figure 1 a)*.

The resonance pulsation for the reduced model is [1] [2] [3]

$$p = \sqrt{\frac{k}{m_1}}\ , \qquad (1)$$

where m_1 is the reduced mass of the system (neglecting the spring mass) and k is the coefficient of stiffness.

The corresponding natural frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m_1}} \qquad (2)$$

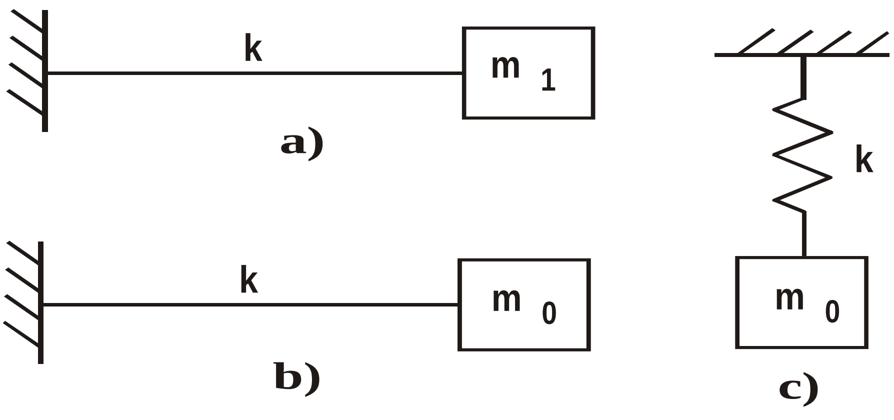

Fig. 1 The calculus models of the 1DOF mechanical system with bending spring distributed mass
a) the system neglecting the spring mass
b) the system with total equivalent mass
c) the equivalent system with coil spring

2. PHYSICAL MODEL OF 1DOF SYSTEM WITH BENDING SPRING DISTRIBUTED MASS

Figure no. 2 shows the calculus diagrams for a simple mechanical system with an bending spring and a single concentrated mass in translation movement of vibration. For simplification, it consider an homogeneous material for the beam and a constant transversal section, therefore the value of the produce EI is constant, where E is the elastic modulus and I is the geometrical moment of inertia of the section.

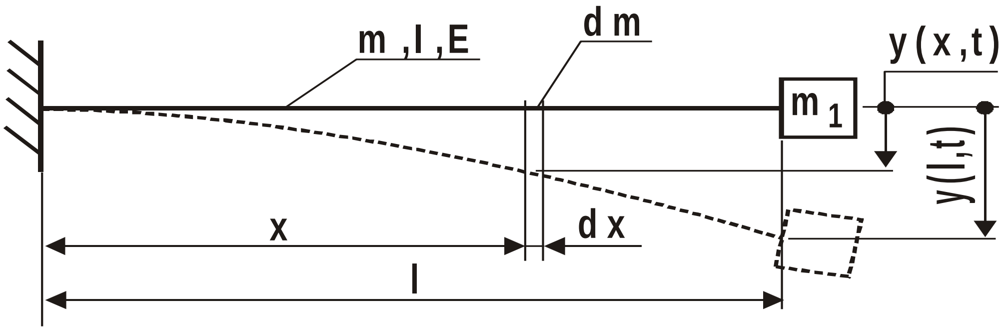

Fig. 2 The calculus diagram for resonant equivalent mass
(neglecting the beam distributed mass on the static deflection)

The linear specific mass of the beam is

$$\rho_l = \frac{m}{l} , \tag{3}$$

the mass of the infinitesimal hatching element being [2] as follows:

$$dm = \rho_l dx = \frac{m}{l} dx \tag{4}$$

The study of the reducing mass of the bending spring can be done in two different hypothesis for the calculus of the static deformation of the beam:
1)taking into consideration only the weight of mass m_1;
2)taking into consideration the total weight of mass m_1 and the mass m of the beam.

3. MATHEMATICAL MODEL OF 1DOF SYSTEM WITH BENDING SPRING DISTRIBUTED MASS

3.1. The model with static deflection done by the mass m_1 only

According to Hooke law, the equation of the medium bending fiber for the model from *figure 2* is acc. to [1]

$$EI \frac{d^2 y}{dx^2}(x,t) = -M(x,t) , \tag{5}$$

where the bending moment of the weight $G_1 = m_1 g$ in section located at distance x from the clamping has the next form:

$$M(x,t) = -m_1 g(l - x) \tag{6}$$

Since the produce EI is constant along the beam, the successive integrations upon variable x of the equation *(5)* lead to the next relations

$$\frac{dy}{dx}(x,t) = \frac{m_1 g}{EI}\left(lx - \frac{x^2}{2}\right) + C_1 \tag{7}$$

and

$$y(x,t) = \frac{m_1 g}{EI}\left(l\frac{x^2}{2} - \frac{x^3}{6}\right) + C_1 x + C_2 \ , \tag{8}$$

where C_1 and C_2 are integration constants.

The values of the constants C_1 and C_2 can be determinate from limit conditions of fixed beam in the clamping (assuming that is a rigid clamping):

$$y(0,t) = 0 \tag{9}$$

$$\varphi(0,t) = \frac{dy}{dx}(0,t) = 0 \tag{10}$$

The condition *(9)* means that the displacement y in the clamping is zero and the condition *(10)* means that the angle of shearing deformation φ is zero.

From the limit conditions, the integrations constants are $C_1 = 0$ and $C_2 = 0$, the formula of medium bending fiber being:

$$y(x,t) = \frac{m_1 g}{EI}\left(l\frac{x^2}{2} - \frac{x^3}{6}\right) \tag{11}$$

The elastic deflection of the beam in the free end m_1 is obtaining for $x = l$:

$$y(l,t) = \frac{m_1 g l^3}{3EI} \tag{12}$$

The deflection of any section of the beam can be wrote in function of the deflection of the free end from the equations *(11)* and *(12)* as follows:

$$y(x,t) = \frac{3lx^2 - x^3}{2l^3} y(l,t) \tag{13}$$

Neglecting the displacements of the beam's sections along the *x* axis, the differential of the relation *(13)* lead to the relation between the velocities of different sections along *y* axis:

$$\dot{y}(x,t) = \frac{3lx^2 - x^3}{2l^3} \dot{y}(l,t) \tag{14}$$

The kinetic energy of the element dm located on the distance x is

$$dE = \frac{dm}{2}\dot{y}^2 = \frac{1}{2}\frac{m}{l}dx[\dot{y}(x,t)]^2 \ , \tag{15}$$

or

$$dE = \frac{1}{2}\frac{m}{l}\left(\frac{3lx^2 - x^3}{2l^3}\right)^2 [\dot{y}(l,t)]^2 \tag{16}$$

The total energy for the entire beam is obtained by integration

$$E = \int dE = \frac{1}{2}\frac{m}{4l^7} \cdot \mathbf{I} \cdot [\dot{y}(l,t)]^2 \ , \tag{17}$$

where the integrale $\mathbf{I}$ is:

$$\mathbf{I}=\int_0^l \left(3lx^2-x^3\right)^2 dx=\left(9l^2\frac{x^5}{5}-6l\frac{x^6}{6}+\frac{x^7}{7}\right)\Bigg|_0^l=\frac{33}{35}l^7 \tag{18}$$

Since the general relation of the kinetic energy of the reduced mass is

$$E=\frac{1}{2}m_r v^2=\frac{1}{2}m_r\left[\dot{y}(l,t)\right]^2 , \tag{19}$$

where m_r is the reduced mass and v is the velocity of the section where the reducing calculus is to be done, through identification of the inertial terms from *(17)* and *(19)* it obtains the expression of the reduced/equivalent mass of the beam on the end of it:

$$m_r\equiv m_{eqv}=\frac{33}{140}m\approx 0{,}2357m \tag{20}$$

The total kinetic energy of the mechanical system is obtaining by summing the energy of the beam and the energy of the mass element m_1

$$E=E_{beam}+E_{m_1}=\frac{1}{2}m_0\left[\dot{y}(l,t)\right]^2 , \tag{21}$$

where m_0 is the total reduced/equivalent mass of the system in the section end of the beam (where is located the mass m_1); the diagram of the equivalent system is shown in *figure 1 b)*.

Through the identification of the terms of the kinetic energies, it obtains expression of the total reduced/equivalent mass:

$$m_0=m_1+\frac{33}{140}m \tag{22}$$

Analyzing the relation *(22)*, it may take the next preliminary conclusions:

-about one quarter of the beam mass is taking into consideration for the calculus of the total kinetic energy of the system;

-the total mass of the system is bigger than the mass m_1, therefore the value of resonance pulsation is decreasing;

-the equivalent calculus diagram is shown in *figure 1 c)*, the resonance pulsation and the natural frequency having the following expressions:

$$p_{eqv}=\sqrt{\frac{k}{m_0}} \tag{23}$$

$$f_{eqv}=\frac{1}{2\pi}\sqrt{\frac{k}{m_0}} \tag{24}$$

3.2. The model with static deflection done by the mass m_1 and the distributed mass of the beam spring

Taking into consideration that the mass of the beam is homogeneous distributed, the linear specific weight of it is

$$q=\frac{mg}{l} , \tag{25}$$

where q is the specific weight and g is the gravity acceleration.

It considers that the statically bending of the beam is produced by the weight of the mass m_1 and the distributed force done by the weight of the beam (see the calculus model from *figure 3*).

The bending moment of the section situated on the distance x from the

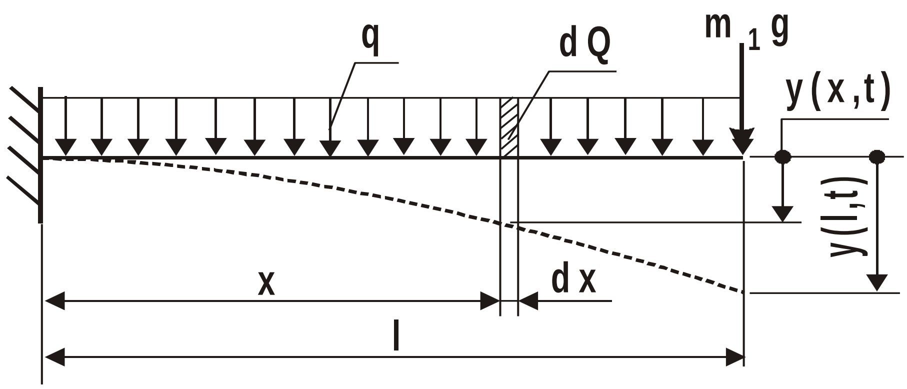

Fig. 3 The calculus diagram for resonant equivalent mass (including the beam distributed mass on the static deflection calculus)

clamping is:

$$M(x,t) = -m_1 g(l-x) - \frac{q}{2}(l-x)^2 \ , \qquad (26)$$

or

$$M(x,t) = -g\left[\left(m_1 + \frac{m}{2}\right)l - (m_1+m)x + \frac{m}{2l}x^2\right] , \qquad (27)$$

where it has considered the homogeneity of the beam and specific weight from *(25)*.

In order to determine the medium bending fiber, it consider the equation *(5)* and the moment *(27)*. Trough successive integrations, it obtains the general relations for the angle of shearing deformation $\varphi(x,t)$ and for displacement $y(x,t)$ as follows:

$$EI\varphi(x,t) = C_1 + \left(m_1 + \frac{m}{2}\right)glx - (m_1+m)g\frac{x^2}{2} + \frac{mg}{6l}x^3 \qquad (28)$$

$$EIy(x,t) = C_2 + C_1 x + \left(m_1 + \frac{m}{2}\right)gl\frac{x^2}{2} - (m_1+m)g\frac{x^3}{6} + \frac{mg}{24l}x^4 \qquad (29)$$

The values of the integration constants C_1 and C_2 are determinate in same way like for *§3.1* and both of them are zero. Thus, the expression for the displacement of the section situated at the distance x from the clamping is:

$$y(x,t) = \frac{gx^2}{2EI}\left[\left(m_1 + \frac{m}{2}\right)l - (m_1+m)\frac{x}{3} + \frac{m}{12l}x^2\right] \qquad (30)$$

The displacement of the free end of the beam is done by the particular condition $x = l$:

$$y(l,t) = \frac{gl^3(8m_1 + 3m)}{24EI} \qquad (31)$$

From the equations *(30)* and *(31)*, it can obtain the relation between the displacements of any section of the beam and the free end of it as follows

$$y(x,t) = A(x,l,m,m_1)\cdot y(l,t) \ , \qquad (32)$$

where $A(x,l,m,m_1)$ is:

$$A = \frac{x^2\left[6(2m_1+m)l^2 - 4(m_1+m)lx + mx^2\right]}{l^4(8m_1+3m)} \qquad (33)$$

Trough derivation, the relation *(32)* between the displacements is available also for the velocities of two sections:

$$\dot{y}(x,t) = A(x,l,m,m_1)\cdot \dot{y}(l,t) \qquad (34)$$

The kinetic energy of the beam element dm from the section x is:

$$dE = \frac{1}{2}dm[\dot{y}(x,t)]^2 = \frac{1}{2}\frac{m}{l}[\dot{y}(x,t)]^2 dx \ , \qquad (35)$$

or

$$dE = \frac{1}{2}\frac{m}{l}[A(x,l,m,m_1)]^2[\dot{y}(l,t)]^2 dx \qquad (36)$$

The kinetic energy of the entire beam (with the free end mass m_1) can be obtained trough integration by the variable x of the relation *(36)*:

$$E = \int dE = \frac{1}{2}\frac{m}{l}[\dot{y}(l,t)]^2 \cdot \int_0^l [A(x,l,m,m_1)]^2 dx \qquad (37)$$

The definite integral from the relation *(37)* can be calculate as follows

$$\int_0^l A^2 dx = \frac{1}{l^8(8m_1+3m)^2}\cdot \mathbf{I} \ , \qquad (38)$$

where

$$\mathbf{I} = \sum_{j=1}^{6} I_j \qquad (39)$$

$$I_1 = \int_0^l 36(2m_1+m)^2 l^4 x^4 dx \qquad (39a)$$

$$I_2 = \int_0^l 16(m_1+m)^2 l^2 x^6 dx \qquad (39b)$$

$$I_3 = \int_0^l m^2 x^8 dx \qquad (39c)$$

$$I_4 = -\int_0^l 48(2m_1+m)(m_1+m)l^3 x^5 dx \qquad (39d)$$

$$I_5 = \int_0^l 12m(2m_1+m)l^2 x^6 dx \qquad (39e)$$

$$I_6 = -\int_0^l 8m(m_1+m)lx^7 dx \qquad (39f)$$

After the calculation of the above six integrals, the expression of $\mathbf{I}$ is

$$\mathbf{I} = \frac{4752m_1^2 + 3717m_1m + 728m^2}{315} l^9 \qquad (40)$$

Thus, the kinetic energy of the beam is similar to *(19)*, where the reduced mass is as follows:

$$m_r = \frac{4752m_1^2 + 3717m_1m + 728m^2}{315(8m_1+3m)^2} m \qquad (41)$$

The total reduced mass m_r can be written

$$m_r = \frac{4752\mu^2 + 3717\mu + 728}{315(8\mu + 3)^2} m = \gamma(\mu) \cdot m \ , \tag{42}$$

where $\mu = m_1 / m$ is the dimensionless mass and $\gamma(\mu)$ is done by the relation:

$$\gamma(\mu) = \frac{4752\mu^2 + 3717\mu + 728}{315(8\mu + 3)^2} \tag{43}$$

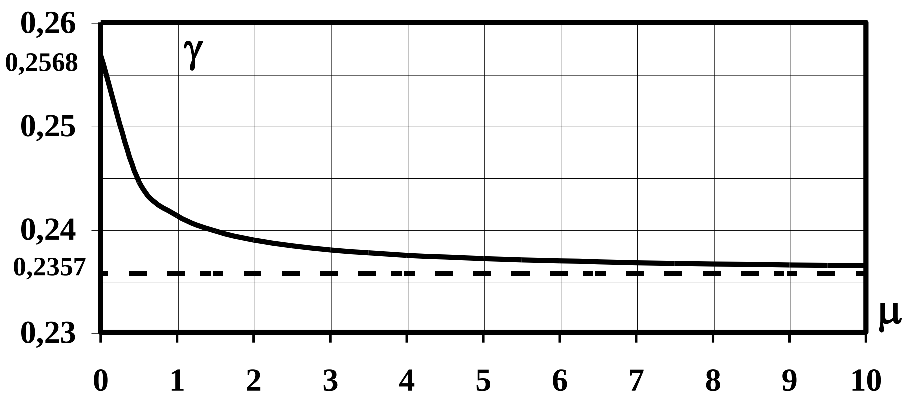

Fig. 4 The variation of the γ parameter depending on "dimensionless mass" μ

The dimensionless parameter $\gamma(\mu)$ is represented in *figure 4*. It can see that the equivalent mass m_r is bigger so much the more the free end mass m_1 is smaller; the maximum of the reduced mass ($m_r \approx 0{,}2568 m$) is obtained when the punctual mass m_1 is zero:

$$\gamma(0) = \frac{104}{405} \approx 0{,}2568 \Rightarrow m_r = \frac{104}{4055} m \tag{44}$$

The influence of the mass m_1 decreases when its value is increasing; theoretically, the increasing of m_1 ad infinitum leads to the minimum value for γ:

$$\min_{0 \le \mu < \infty} (\gamma) = \lim_{\mu \to \infty} \gamma(\mu) = \frac{33}{140} \approx 0{,}2357 \tag{45}$$

Consequently the minimum value for the reduced mass m_r is:

$$m_{r\,min} = m_r(\infty) = \frac{33}{140} m \approx 0{,}2357 m \tag{46}$$

The total kinetic energy of the mechanical system is obtaining by summing the energy of the beam and the energy of the mass element m_1

$$E = E_{beam} + E_{m_1} = \frac{1}{2} m_0 [\dot{y}(l,t)]^2 \ , \tag{47}$$

where m_0 is the total reduced/equivalent mass of the system in the section end of the beam (where is located the mass m_1); the diagram of the equivalent system is shown in *figure 1 b)*.

Through the identification of the terms of the kinetic energies, it obtains expression of the total reduced/equivalent mass:

$$m_0 = m_1 + \frac{4752 m_1^2 + 3717 m_1 m + 728 m^2}{315(8m_1 + 3m)^2} m \qquad (48)$$

Analyzing the relations *(42)*-*(48)*, it may take the next partial conclusions:
-only one fraction of the beam mass is taking into consideration for the calculus of the total kinetic energy of the system;
-the fraction of beam mass which participates into the calculus of the equivalent mass depends on the mass m_1; this fraction is described by the parameter γ which is bigger if the mass of the end of the beam is smaller and inversely
-the maximum influence of the beam mass is reached in absence of the mass m_1;
-the influence of the beam mass is decreased if the value of mass m_1 is very big.

4. CONCLUSIONS

The physical and the calculus model of the mechanical system with an bending beam and a concentrated mass on the end of it make evident the influence of the distributed mass of the spring on the resonance behavior. This influence is higher how much more the beam mass is bigger and the end mass is smaller.

In any case, the equivalent mass of the system is bigger than the concentrated mass m_1, therefore the value of resonance pulsation is decreasing [2]. The modification of the calculus mass and of the resonance frequency is very important especially for the machines and equipments with the resonant function regime.

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