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BOUNDARY CONDITIONS IN THE FORM OF LOADING, ANALYZED WITH FINITE-DIFFERENCE METHOD

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Abstract**—**This paper presents a finite-difference computational method, which describes the plane state of stress of the orthotropic materials. By using a function of the displacement potential, the determination of boundary condition as load for the vertical side on the left is exemplified. The convex corner case is treated separately, when the corner displacement is free and when it is prevented or prescribed. When displacement is imposed in both directions, we analyze the situation in which only one side is leaning, and when both sides are embedded. Depending on the number of points inside, on the sides and the number of convex and concave corners, the number of points on the imaginary boundary and inside the domain under analysis can be established.

Keywords**—**displacement potential function, finite-difference method, orthotropic materials, stress analysis

I. INTRODUCTION

FOR the integration differential equation with partial derivates which describe the plane state of displacement or stress of the anisotropic, orthotropic and isotropic materials, a computational method with finite difference can be applied [1]–[6].

The problem can be expressed in stress leading to Airy function [3], [4]. The second order partial derivatives of this describe the stress field. With the help of the stresses and with the equations of the material the specific strains, respectively displacement can be determined [7], [8].

The disadvantage of using the Airy function is that all the boundary conditions must be given in stresses, because the displacement cannot be expressed in a direct way.

By analogy with the Airy function a „potential function" of the displacement can be used, which makes the prescription of the mixed boundary conditions possible [3]–[4], [6]. The partial derivatives of this function give the displacement in the direction of the coordinate axes. The derivatives of the displacement, namely the derivatives of superior order of the function of the displacement give the specific strains, and through the application of the material equations, these derivatives of superior order will lead to the stress field. This points to the fact that the description of the boundary conditions under the form of prescribed stresses by the load distribution on the boundary becomes possible, because there is a direct relation with differential equations between the displacements and stresses [3], [6]. These relations are estimated with finite differences.

The disadvantage of the method is that we can have body forces only in one direction.

II. FORMULATION WITH FINITE DIFFERENCE SOLUTION

Starting from the idea of Airy stress function, we suppose that in the case of orthotropic materials there is a function $\mathbf{\Psi(x,y)}$. Its partial derivatives give the projections of the displacement. Using Hooke's law, we transcribe the equilibrium equations in specific strains (considering the body force in $\mathbf{x}$-direction $\mathbf{f_x = 0}$), then with the geometrical equations, we obtain [3]:

$$\beta_1 \frac{\partial^4 \Psi}{\partial x^4} + \beta_2 \frac{\partial^4 \Psi}{\partial x^3 \partial y} + \beta_3 \frac{\partial^4 \Psi}{\partial x^2 \partial y^2} + \\ + \beta_4 \frac{\partial^4 \Psi}{\partial x \partial y^3} + \beta_5 \cdot \frac{\partial^4 \Psi}{\partial y^4} = \beta_0 \cdot f_y, \tag{1}$$

where β_i are the coefficients of the differential equation of Ψ, and $\mathbf{f_y}$ is the body force in $\mathbf{y}$ - direction.

The solution of this is the potential function sought by us.

If the partial derivatives of the (1) are replaced by centered finite differences, we achieve the molecules presented in Fig. 1., where $\mathbf{h}$, $\mathbf{k}$ is grid lengths in the $\mathbf{x}$ - and $\mathbf{y}$ - directions.

	$-2\cdot h$	$-h$	0	h	$2\cdot h$
$2\cdot k$		$-\frac{\beta_4}{4\cdot h\cdot k^3}$	$\frac{\beta_5}{k^4}$	$\frac{\beta_4}{4\cdot h\cdot k^3}$	
k	$-\frac{\beta_2}{4\cdot h^3\cdot k}$	$\frac{\beta_2}{2\cdot h^3\cdot k}+\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}$	$-2\cdot\frac{\beta_3}{h^2\cdot k^2}-4\cdot\frac{\beta_5}{k^4}$	$-\frac{\beta_2}{2\cdot h^3\cdot k}+\frac{\beta_3}{h^2\cdot k^2}-\frac{\beta_4}{2\cdot h\cdot k^3}$	$\frac{\beta_2}{4\cdot h^3\cdot k}$
0	$\frac{\beta_1}{h^4}$	$-4\cdot\frac{\beta_1}{h^4}-2\cdot\frac{\beta_3}{h^2\cdot k^2}$	$6\cdot\frac{\beta_1}{h^4}+4\cdot\frac{\beta_3}{h^2\cdot k^2}+6\cdot\frac{\beta_5}{k^4}$	$-4\cdot\frac{\beta_1}{h^4}-2\cdot\frac{\beta_3}{h^2\cdot k^2}$	$\frac{\beta_1}{h^4}$
$-k$	$\frac{\beta_2}{4\cdot h^3\cdot k}$	$-\frac{\beta_2}{2\cdot h^3\cdot k}+\frac{\beta_3}{h^2\cdot k^2}-\frac{\beta_4}{2\cdot h\cdot k^3}$	$-2\cdot\frac{\beta_3}{h^2\cdot k^2}-4\cdot\frac{\beta_5}{k^4}$	$\frac{\beta_2}{2\cdot h^3\cdot k}+\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}$	$-\frac{\beta_2}{4\cdot h^3\cdot k}$
$-2\cdot k$		$\frac{\beta_4}{4\cdot h\cdot k^3}$	$\frac{\beta_5}{k^4}$	$-\frac{\beta_4}{4\cdot h\cdot k^3}$	

Fig. 1. The approximation with finite-difference method of the (1).

Therefore, in the point of coordinates (x,y), which is the point (i,j) of the grid for calculating finite differences, we can write the following equation:

$$\begin{aligned}
&\frac{\beta_4}{4\cdot h\cdot k^3}\cdot\Psi(i-1,j-2)+\frac{\beta_5}{k^4}\cdot\Psi(i,j-2)-\\
&-\frac{\beta_4}{4\cdot h\cdot k^3}\cdot\Psi(i+1,j-2)+\frac{\beta_2}{4\cdot h^3\cdot k}\cdot\Psi(i-2,j-1)-\\
&-\left(\frac{\beta_2}{2\cdot h^3\cdot k}-\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}\right)\cdot\Psi(i-1,j-1)-\\
&-\left(2\cdot\frac{\beta_3}{h^2\cdot k^2}+4\cdot\frac{\beta_5}{k^4}\right)\cdot\Psi(i,j-1)+\\
&+\left(\frac{\beta_2}{2\cdot h^3\cdot k}+\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}\right)\cdot\Psi(i+1,j-1)-\\
&-\frac{\beta_2}{4\cdot h^3\cdot k}\cdot\Psi(i+2,j-1)+\frac{\beta_1}{h^4}\cdot\Psi(i-2,j)-\\
&-\left(4\cdot\frac{\beta_1}{h^4}+2\cdot\frac{\beta_3}{h^2\cdot k^2}\right)\cdot\Psi(i-1,j)+\\
&+\left(6\cdot\frac{\beta_1}{h^4}+4\cdot\frac{\beta_3}{h^2\cdot k^2}+6\cdot\frac{\beta_5}{k^4}\right)\cdot\Psi(i,j)-\\
&-\left(4\cdot\frac{\beta_1}{h^4}+2\cdot\frac{\beta_3}{h^2\cdot k^2}\right)\cdot\Psi(i+1,j)+\\
&+\frac{\beta_1}{h^4}\cdot\Psi(i+2,j)-\frac{\beta_2}{4\cdot h^3\cdot k}\cdot\Psi(i-2,j+1)+\\
&+\left(\frac{\beta_2}{2\cdot h^3\cdot k}+\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}\right)\cdot\Psi(i-1,j+1)-\\
&-\left(2\cdot\frac{\beta_3}{h^2\cdot k^2}+4\cdot\frac{\beta_5}{k^4}\right)\cdot\Psi(i,j+1)-\\
&-\left(\frac{\beta_2}{2\cdot h^3\cdot k}-\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}\right)\cdot\Psi(i+1,j+1)+\\
&+\frac{\beta_2}{4\cdot h^3\cdot k}\cdot\Psi(i+2,j+1)-\\
&-\frac{\beta_4}{4\cdot h\cdot k^3}\cdot f(i-1,j+2)+\frac{\beta_5}{k^4}\cdot\Psi(i,j+2)+\\
&-\frac{\beta_4}{4\cdot h\cdot k^3}\cdot\Psi(i,j+2)=-f_y(i,j).
\end{aligned}\qquad(2)$$

In these equations the values of Ψ function appear, taken in the neighboring points, resulting in a system of equations to be solved in the $\Psi(i,j)$ nodal values.

For the boundary points we can't apply the molecules from Fig. 1., because the values of Ψ appear in (2) in some non-existing external nodes. These values also appear when we apply (2) for the nodes next to the boundary ones: these external nodes define a new imaginary boundary beyond the physical one, increasing the number of the unknowns to be determined (Fig. 2.).

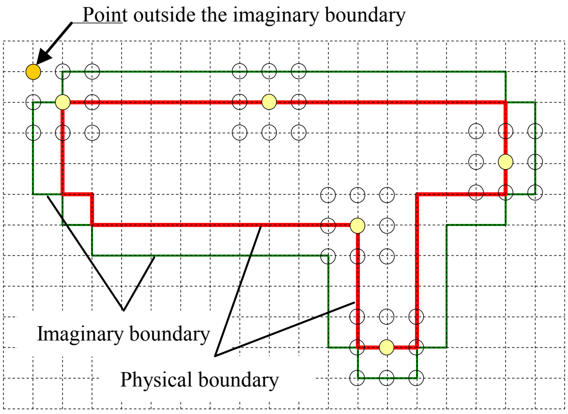

Fig. 2. The imaginary boundary that appears due to finite-difference approximation.

The system of equations can be solved only by writing boundary conditions: we will give these conditions in all boundary nodes, as prescribed displacements and/or loading forces.

III. THE BOUNDARY CONDITIONS IN THE FORM OF LOADING

Distributed stress that loads the boundary is defined by its projections according to x and y directions, noted as p_x and p_y. This stress is generally described by an arbitrary function. During the mesh, this function is replaced by a step function.

If we cut an element from the area around a boundary point, the stresses along the boundary must be in equilibrium with the external loads. Therefore we can write the relations which equal the projections of the exterior load with the stresses along the boundary, and the boundary conditions are written by stresses with the help of Ψ function derivates, rewriting these derivates with finite-differences (Fig. 3.).

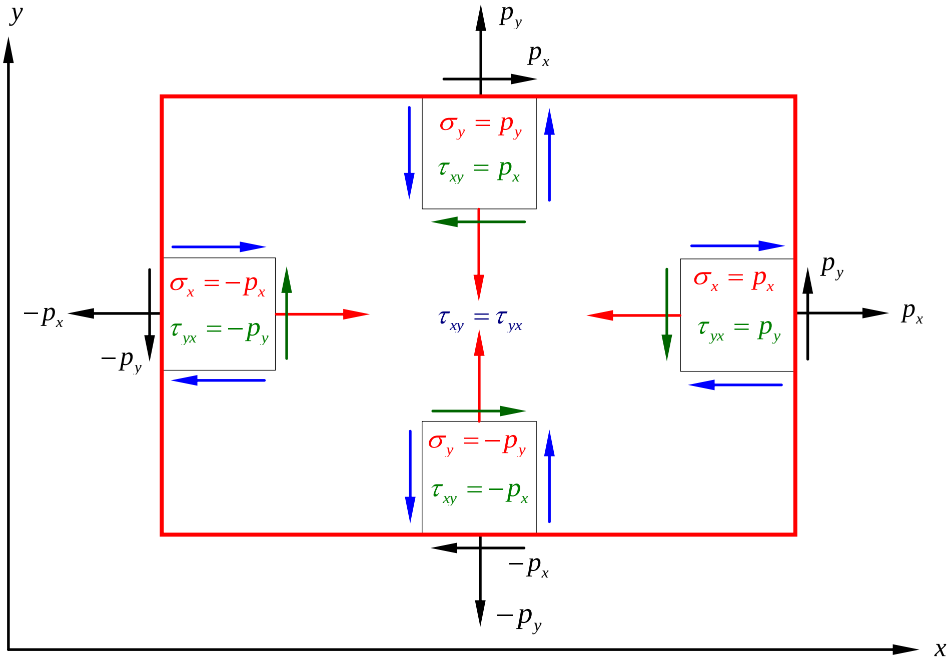

Fig. 3. Stresses along the boundary.

Relations between specific stresses and strains are given by the generalized Hooke's law, the specific deformations and displacements of geometrical equations and the displacements and Ψ function by (3).

$$\mathbf{u} = \alpha_1 \cdot \frac{\partial^2 \Psi}{\partial x^2} + \alpha_2 \cdot \frac{\partial^2 \Psi}{\partial x \cdot \partial y} + \alpha_3 \cdot \frac{\partial^2 \Psi}{\partial y^2}, \qquad \mathbf{v} = \alpha_4 \cdot \frac{\partial^2 \Psi}{\partial x^2} + \alpha_5 \cdot \frac{\partial^2 \Psi}{\partial x \cdot \partial y} + \alpha_6 \cdot \frac{\partial^2 \Psi}{\partial y^2}, \tag{3}$$

where, $\mathbf{u}$, $\mathbf{v}$ are displacement components in the $\mathbf{x}$ - and $\mathbf{y}$ - directions, α_i are coefficients of the derivates of Ψ.

Thus [4], we obtain the relation:

$$\begin{aligned}
\sigma_x &= E_{11} \cdot \varepsilon_x + E_{12} \cdot \varepsilon_y + E_{13} \cdot \gamma_{xy} = \\
&= E_{11} \cdot \frac{\partial u}{\partial x} + E_{12} \cdot \frac{\partial v}{\partial y} + E_{13} \cdot \frac{\partial u}{\partial y} + E_{13} \cdot \frac{\partial v}{\partial x} = \\
&= E_{11} \cdot \left(\alpha_1 \cdot \frac{\partial^3 \Psi}{\partial x^3} + \alpha_2 \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_3 \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} \right) + \\
&+ E_{12} \cdot \left(\alpha_4 \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_5 \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} + \alpha_6 \cdot \frac{\partial^3 \Psi}{\partial y^3} \right) + \\
&+ E_{13} \cdot \left(\alpha_1 \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_2 \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} + \alpha_3 \cdot \frac{\partial^3 \Psi}{\partial y^3} \right) + \\
&+ E_{13} \cdot \left(\alpha_4 \cdot \frac{\partial^3 \Psi}{\partial x^3} + \alpha_5 \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \alpha_6 \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^3} \right) = \\
&= (E_{11} \cdot \alpha_1 + E_{13} \cdot \alpha_4) \cdot \frac{\partial^3 \Psi}{\partial x^3} + \\
&+ (E_{11} \cdot \alpha_2 + E_{12} \cdot \alpha_4 + E_{13} \cdot \alpha_1 + E_{13} \cdot \alpha_5) \cdot \frac{\partial^3 \Psi}{\partial x^2 \cdot \partial y} + \\
&+ (E_{11} \cdot \alpha_3 + E_{12} \cdot \alpha_5 + E_{13} \cdot \alpha_2 + E_{13} \cdot \alpha_6) \cdot \frac{\partial^3 \Psi}{\partial x \cdot \partial y^2} + \\
&+ (E_{12} \cdot \alpha_6 + E_{13} \cdot \alpha_3) \cdot \frac{\partial^3 \Psi}{\partial y^3},
\end{aligned} \tag{4}$$

where σ_x is the normal stress, E_{ij} are the coefficients of the elasticity matrix.

The boundary conditions are written by stresses with the help of Ψ function derivates, rewriting these derivates with finite-differences.

We exemplify determining conditions of the boundary for the vertical side on the left, as follows:

$$\begin{aligned}
&-\frac{c_4}{2 \cdot k^3} \cdot \Psi(j-2, i) - \\
&-\left(\frac{c_2}{2 \cdot h^2 \cdot k} + \frac{c_3}{2 \cdot h \cdot k^2} \right) \cdot \Psi(j-1, i-1) + \\
&+\left(\frac{c_2}{h^2 \cdot k} + \frac{c_4}{k^3} \right) \cdot \Psi(j-1, i) - \\
&-\left(\frac{c_2}{2 \cdot h^2 \cdot k} - \frac{c_3}{2 \cdot h \cdot k^2} \right) \cdot \Psi(j-1, i+1) +
\end{aligned}$$

$$\begin{aligned}&+\frac{c_3}{hk^2}\cdot\Psi(j,i-1)-\frac{c_1}{h^3}\cdot\Psi(j,i)+\\&+\left(\frac{3\cdot c_1}{h^3}-\frac{c_3}{hk^2}\right)\cdot\Psi(j,i+1)-\\&-\frac{3\cdot c_1}{h^3}\cdot\Psi(j,i+2)+\frac{c_1}{h^3}\cdot\Psi(j,i+3)+\\&+\left(\frac{c_2}{2h^2k}-\frac{c_3}{2hk^2}\right)\cdot\Psi(j+1,i-1)-\\&-\left(\frac{c_2}{h^2k}+\frac{c_4}{k^3}\right)\cdot\Psi(j+1,i)+\\&+\left(\frac{c_2}{2h^2k}+\frac{c_3}{2hk^2}\right)\cdot\Psi(j+1,i+1)+\\&+\frac{c_4}{2k^3}\cdot\Psi(j+2,i)=-p_x,\end{aligned}\tag{5}$$

where,

$$\begin{aligned}c_1&=E_{11}\cdot\alpha_1+E_{13}\cdot\alpha_4,\\c_2&=E_{11}\cdot\alpha_2+E_{12}\cdot\alpha_4+E_{13}\cdot\alpha_1+E_{13}\cdot\alpha_5,\\c_3&=E_{11}\cdot\alpha_3+E_{12}\cdot\alpha_5+E_{13}\cdot\alpha_2+E_{13}\cdot\alpha_6,\\c_4&=E_{12}\cdot\alpha_6+E_{13}\cdot\alpha_3.\end{aligned}\tag{6}$$

We can observe that if we apply the molecule for a grid node positioned on the boundary, it will be based on three points that are on the imaginary boundary.

The concave corners will not raise issues or difficulties, however in the convex corners this scheme would include a point that does not belong to the imaginary boundary (Fig. 2.).

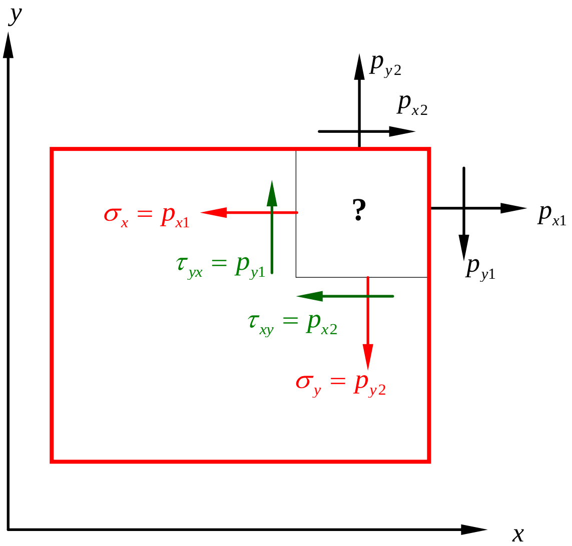

Fig. 4. Incompatible boundary conditions in convex corner.

Writing the boundary conditions as prescribed loads on the convex corners is an issue that should be considered separately. If we have tangential load (Fig. 4.) along the sides of boundaries the stress will be $\tau_{xy}=p_{x2}$ and $\tau_{yx}=p_{y1}$. When $p_{x2}\neq p_{y1}$ and/or their directions do not correspond to the principle of duality of tangential stresses, we have to redistribute the loads in these corners in order to solve this problem.

This redistribution is arbitrary. If the displacement of the corner is free in both directions, then the tangential stresses can be calculated with:

$$p'_{y1}=p'_{x2}=\frac{p_{y1}+p_{x2}}{2}\quad\Rightarrow\quad\tau_{xy}=\tau_{yx},\tag{7.a}$$

and

$$\begin{aligned}p'_{x1}&=p_{x1}+\left(p_{x2}-\frac{p_{y1}+p_{x2}}{2}\right)=\\&=p_{x1}+\frac{p_{x2}-p_{y1}}{2}\Rightarrow\sigma_x,\end{aligned}\tag{7.b}$$

$$\begin{aligned}p'_{y2}&=p_{y2}+\left(p_{y1}-\frac{p_{y1}+p_{x2}}{2}\right)=\\&=p_{y2}+\frac{p_{y1}-p_{x2}}{2}\Rightarrow\sigma_y.\end{aligned}\tag{7.c}$$

This recalculation assures the principle of duality and preserves the resulted value of the load, $p'_{x1}+p'_{x2}=p_{x1}+p_{x2}$ and $p'_{y1}+p'_{y2}=p_{y1}+p_{y2}$.

If the displacement of the corner point is prevented or prescribed in one direction, then the tangential load in the grid node has to be eliminated.

It is considered that the supports are ideal, without friction, so in the direction of free displacement there is no tangential force. According to the principle of duality, in this case there can be no tangential force in the perpendicular direction to it (Fig. 5.). Prescribing null value of tangential stresses is the third equation which is written for those points.

In practice, there are two possibilities at corner points, where we imposed displacements in both directions:

1. If only at least one side is leaning: the third condition of boundary results because of lack of friction as null value of the tangential stress.

2. If both sides are embedded: the corner's turning point is blocked, so we can accept the null tangential tension ($\gamma=0\Rightarrow\tau=0$).

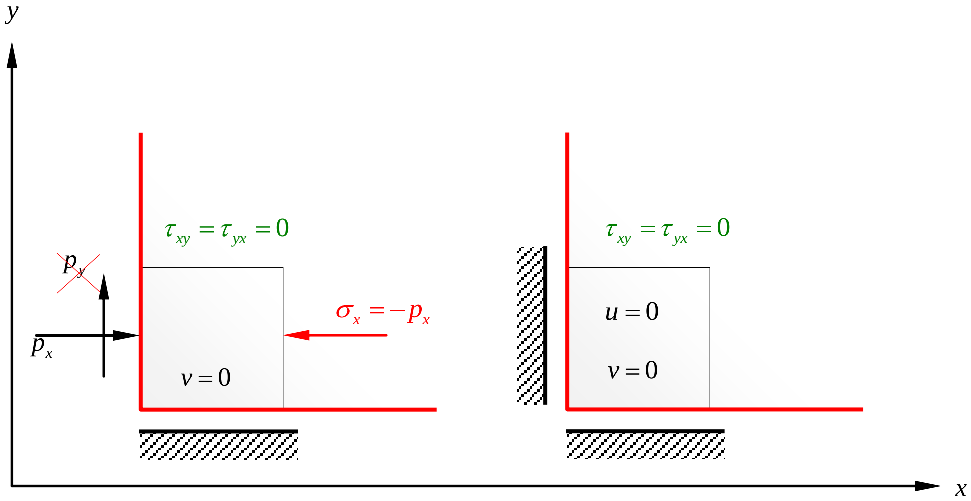

Fig. 5. Analyze of concave corner point.

Points of concave corners must be treated separately. Although there are no fictitious points, the normal of the contour is not defined, and we cannot separate one element of a rectangle the equilibrium of which we could analyze. The standard procedure would be rounding the corner points, that is to fill the corner with a triangular element and to analyze its equilibrium state (Fig. 6.).

Moving the point load on the hypotenuse of the corner:

$$\mathbf{F_x = p_{x1} \cdot k + p_{x2} \cdot h,} \quad \mathbf{F_y = p_{y1} \cdot k + p_{y2} \cdot h.} \tag{8}$$

Equilibrium equations written for triangle lead to tensions on the legs. This problem has a unique solution because we cannot define the three stresses of the equilibrium equations written on the hypotenuse load projections. Therefore, the average tangential load is calculated according to (7.a), which gives the corresponding sign of the tangential stress. The difference between initial and average value is added to the normal load (7.b and c), the sum being equal to the normal stress.

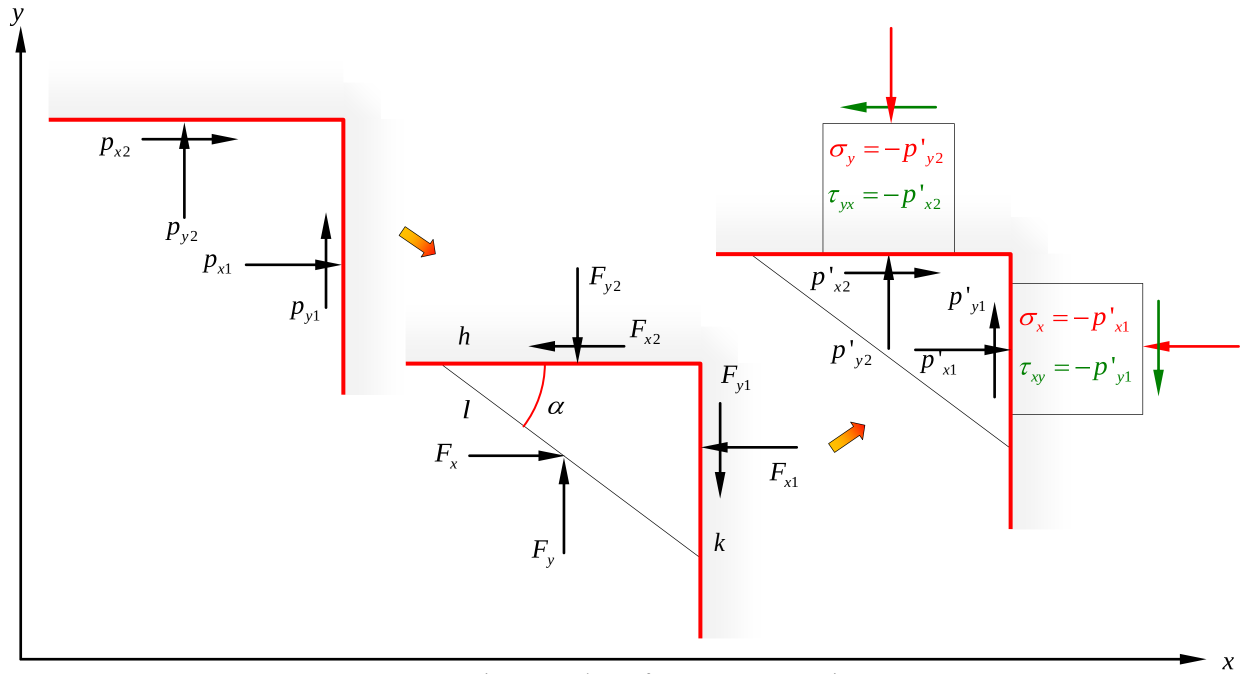

Fig. 6. Analyze of concave corner points.