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NONLINEAR THIRD ORDER DAMPED FORCED VIBRATIONS

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***Abstract*—**In this paper we discuss the forced vibrations of a single degree of freedom system with nonlinear arbitrary integer order damping and with linear stiffness. The method used for our theoretical study is the multiple scales one and the complete calculation is performed for the case of the third order damped system. For this mechanical system the authors obtained the analytical characterization of the motion around and a simply stable equilibrium position for the unexcited system. A special discussion is reserved for the steady-state motion. Finally, a numerical analysis is performed using the Runge–Kutta fourth order method and the theoretical results are compared with those obtained by numerical simulation, a good agreement between them being found. The condition found for the simply stable equilibrium position is in this case identical to that for the stable motion of the system.

***Keywords*—**cubic damper, multiple scales method, nonlinear, numerical simulation

I. INTRODUCTION

THE study of the nonlinear vibrations is still an open field because of the complications which appear in the mathematical treatment of the differential equations that arise.

The exact solution of such equations can be found only in some particular case and it is, generally speaking, represented by certain special functions.

Usually, the authors prefer an approximate study using different methods derived from the perturbation techniques. One of these methods is the method of multiple scales. The idea of this method is to write the solution $\mathbf{x} = \mathbf{x}(\mathbf{t})$ of the differential equation as a sum of terms of the form $\boldsymbol{\varepsilon}^{\mathbf{i}}\mathbf{x}_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{N}$, each component being a function of various scales of time, $\mathbf{x}_{\mathbf{i}} = \mathbf{x}_{\mathbf{i}}(\mathbf{T}_0, \mathbf{T}_1, \ldots)$, with $\mathbf{T}_{\mathbf{j}} = \boldsymbol{\varepsilon}^{\mathbf{j}}\mathbf{t}$, $\mathbf{j} \in \mathbb{N}$, $\boldsymbol{\varepsilon}$ being a small real number.

II. MECHANICAL MODEL

The mechanical model is drawn in Fig. 1. and it consists in the mass $\mathbf{m}$ linked to a fixed support by a linear spring of stiffness $\mathbf{k}$ and by a damper for which the damping force $\mathbf{F_d}$ is [4], [5], [6]

$$\mathbf{F_d} = \mathbf{c_1}\dot{\mathbf{x}} + \mathbf{c_p}\dot{\mathbf{x}}^{\mathbf{p}}, \tag{1}$$

where $\dot{\mathbf{x}}$ is the velocity of the mass $\mathbf{m}$, $\mathbf{c_1}$ and $\mathbf{c_p}$ are damping coefficients having the dimensions $\mathrm{Ns/m}$ and $\mathrm{Ns}^p/\mathrm{m}^p$, $\mathbf{p}$ is a positive integer power greater than unity, $\mathbf{x}$ being the displacement.

The mass $\mathbf{m}$ is excited by the harmonic force $\mathbf{F}$ given by the expression

$$\mathbf{F} = \mathbf{F_0}\cos(\boldsymbol{\omega}\mathbf{t}). \tag{2}$$

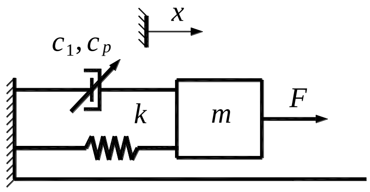

Fig. 1. Mechanical model.

Isolating the body of mass $\mathbf{m}$, one obtains the law of motion

$$\mathbf{m}\ddot{\mathbf{x}} = -\mathbf{k}\mathbf{x} - \mathbf{c_1}\dot{\mathbf{x}} - \mathbf{c_p}\dot{\mathbf{x}}^{\mathbf{p}} + \mathbf{F}, \tag{3}$$

wherefrom it results

$$\ddot{\mathbf{x}} + \frac{\mathbf{k}}{\mathbf{m}}\mathbf{x} = -\frac{\mathbf{c_1}}{\mathbf{m}}\dot{\mathbf{x}} - \frac{\mathbf{c_p}}{\mathbf{m}}\dot{\mathbf{x}}^{\mathbf{p}} + \frac{\mathbf{F_0}}{\mathbf{m}}\cos(\boldsymbol{\omega}\mathbf{t}). \tag{4}$$

Denoting

$$\frac{\mathbf{k}}{\mathbf{m}} = \boldsymbol{\omega}_0^2, \tag{5}$$

$$\frac{\mathbf{c_1}}{\mathbf{m}} = 2\boldsymbol{\varepsilon}\boldsymbol{\zeta}_1, \tag{6}$$

$$\frac{c_p}{m} = \varepsilon\zeta_p, \tag{7}$$

$$\frac{F_0}{m} = \varepsilon f, \tag{8}$$

equation (4) becomes

$$\ddot{x} + \omega_0^2 x = -2\varepsilon\varepsilon_1\dot{x} - \varepsilon\zeta_p\dot{x}^p + \varepsilon f\cos\omega t, \tag{9}$$

where ε is a small and dimensionless parameter, necessary to apply the multiple scale method.

III. ANALYTICAL APPROACH

We will consider that the solution of the differential equation (4) may be put in the form [1], [3], [7], [8]

$$x = x_0(T_0, T_1, T_2, \ldots) + \varepsilon x_1(T_0, T_1, T_2, \ldots) + \varepsilon^2 x_2(T_0, T_1, T_2, \ldots) + \ldots, \tag{10}$$

where

$$T_i = \varepsilon^i t,\ i = 0, 1, \ldots \tag{11}$$

It results

$$\frac{dx}{dt} = \frac{\partial x}{\partial T_0}\frac{dT_0}{dt} + \frac{\partial x}{\partial T_1}\frac{dT_1}{dt} + \frac{\partial x}{\partial T_2}\frac{dT_2}{dt} + \ldots, \tag{12}$$

i.e.

$$\frac{dx}{dt} = \frac{\partial x}{\partial T_0} + \varepsilon\frac{\partial x}{\partial T_1} + \varepsilon^2\frac{\partial x}{\partial T_2} + \ldots, \tag{13}$$

so that

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{\partial x}{\partial T_0} + \varepsilon\frac{\partial x}{\partial T_1} + \varepsilon^2\frac{\partial x}{\partial T_2} + \ldots\right) = \\ &\frac{\partial^2 x}{\partial T_0^2} + \varepsilon\frac{\partial^2 x}{\partial T_0\partial T_1} + \varepsilon^2\frac{\partial^2 x}{\partial T_0\partial T_2} + \ldots + \\ &\varepsilon\frac{\partial^2 x}{\partial T_1\partial T_0} + \varepsilon^2\frac{\partial^2 x}{\partial T_1^2} + \varepsilon^3\frac{\partial^2 x}{\partial T_1\partial T_2} + \ldots + \\ &\varepsilon^2\frac{\partial^2 x}{\partial T_2\partial T_0} + \varepsilon^3\frac{\partial^2 x}{\partial T_2\partial T_1} + \varepsilon^4\frac{\partial^2 x}{\partial T_2^2} + \ldots\end{aligned} \tag{14}$$

On the other hand,

$$\frac{\partial^2 x}{\partial T_0^2} = \frac{\partial^2 x_0}{\partial T_0^2} + \varepsilon\frac{\partial^2 x_1}{\partial T_0^2} + \varepsilon^2\frac{\partial^2 x_2}{\partial T_0^2} + \ldots, \tag{15}$$

$$\frac{\partial^2 x}{\partial T_0\partial T_1} = \frac{\partial^2 x_0}{\partial T_0\partial T_1} + \varepsilon\frac{\partial^2 x_1}{\partial T_0\partial T_1} + \varepsilon^2\frac{\partial^2 x_2}{\partial T_0\partial T_1} + \ldots, \tag{16}$$

and so on, and from here we deduce that (14) may be put in the form

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial T_0^2} + 2\varepsilon\left(\frac{\partial^2 x_0}{\partial T_0\partial T_1} + \varepsilon\frac{\partial^2 x_1}{\partial T_0\partial T_1} + \varepsilon^2\frac{\partial^2 x_1}{\partial T_0\partial T_1}\right) + \ldots \tag{17}$$

Moreover

$$\dot{x}^p = \left(\frac{\partial x}{\partial T_0} + \varepsilon\frac{\partial x}{\partial T_1} + \varepsilon^2\frac{\partial x}{\partial T_2} + \ldots\right)^p = \left(\frac{\partial x}{\partial T_0}\right)^p + \varepsilon p\left(\frac{\partial x}{\partial T_0}\right)^{p-1}\frac{\partial x}{\partial T_1} + \ldots \tag{18}$$

and since

$$\frac{\partial x}{\partial T_0} = \frac{\partial x_0}{\partial T_0} + \varepsilon\frac{\partial x_1}{\partial T_0} + \varepsilon^2\frac{\partial x_2}{\partial T_0} + \ldots, \tag{19}$$

$$\left(\frac{\partial x}{\partial T_0}\right)^{p-1} = \left(\frac{\partial x_0}{\partial T_0}\right)^{p-1} + \varepsilon(p-1)\frac{\partial x_1}{\partial T_1} + \ldots, \tag{20}$$

$$\left(\frac{\partial x}{\partial T_0}\right)^{p} = \left(\frac{\partial x_0}{\partial T_0}\right)^{p} + \varepsilon p\frac{\partial x_1}{\partial T_1} + \ldots \tag{21}$$

by equating the coefficients of different powers of ε in (9), we get

$$\frac{\partial^2 x_0}{\partial T_0^2} + \omega_0^2 x_0 = 0, \tag{22}$$

for ε^0, respectively

$$\frac{\partial^2 x_1}{\partial T_0^2} + \omega_0^2 x_1 = -2\frac{\partial^2 x_0}{\partial T_0\partial T_1} - 2\zeta_1\frac{\partial x_0}{\partial T_0} - \zeta_p\left(\frac{\partial x}{\partial T_0}\right)^p + f\cos(\omega t), \tag{23}$$

for ε^1.

Writing now

$$\omega t = \omega_0 T_0 + \alpha T_1, \tag{24}$$

equation (23) takes the form

$$\frac{\partial^2 x_1}{\partial T_0^2} + \omega_0^2 x_1 = -2\frac{\partial^2 x_0}{\partial T_0 \partial T_1} - 2\zeta_1 \frac{\partial x_0}{\partial T_0} - \zeta_p \left(\frac{\partial x}{\partial T_0}\right)^p + f\cos(\omega_0 T_0 + \alpha T_1). \quad (25)$$

The solution of (22) has the expression

$$x_0 = A(T_1)e^{i\omega_0 T_0} + \overline{A}(T_1)e^{-i\omega_0 T_0}, \quad (26)$$

where the over-bar signifies the complex conjugate.
It results

$$\frac{\partial x_0}{\partial T_0} = i\omega_0 A(T_1)e^{i\omega_0 T_0} - i\omega_0 \overline{A}(T_1)e^{-i\omega_0 T_0}, \quad (27)$$

$$\frac{\partial^2 x_0}{\partial T_0 \partial T_1} = i\omega_0 \frac{\partial A(T_1)}{\partial T_1} e^{i\omega_0 T_0} - i\omega_0 \frac{\partial \overline{A}(T_1)}{\partial T_1} e^{-i\omega_0 T_0}. \quad (28)$$

For the simplification of the calculation, we will limit ourselves to the case $p = 3$, for which

$$\left(\frac{\partial x}{\partial T_0}\right)^3 = \left[i\omega_0 A(T_1)e^{i\omega_0 T_0} - i\omega_0 \overline{A}(T_1)e^{-i\omega_0 T_0}\right]^3 = - i\omega_0^3 A^3 e^{3i\omega_0 T_0} + 3i\omega_0^3 A^2 \overline{A} e^{i\omega_0 T_0} - 3i\omega_0^3 \overline{A}^2 A e^{-i\omega_0 T_0} + i\omega_0^3 \overline{A}^3 e^{-3i\omega_0 T_0}, \quad (29)$$

$$f\cos(\omega_0 T_0 + \alpha T_1) = \frac{1}{2} f\left[e^{i(\omega_0 T_0 + \alpha T_1)} + e^{-i(\omega_0 T_0 + \alpha T_1)}\right] \quad (30)$$

and (25) becomes

$$\frac{\partial^2 x_1}{\partial T_0^2} + \omega_0^2 x_1 = -\left(2i\omega_0 \frac{\partial A}{\partial T_1} e^{i\omega_0 T_0} - 2i\omega_0 \frac{\partial \overline{A}}{\partial T_1} e^{-i\omega_0 T_0}\right) - 2\zeta_1\left(i\omega_0 A e^{i\omega_0 T_0} - i\omega_0 \overline{A} e^{-i\omega_0 T_0}\right) - \zeta_p\left[- i\omega_0^3 A^3 e^{3i\omega_0 T_0} + 3i\omega_0^3 A^2 \overline{A} e^{i\omega_0 T_0} - 3i\omega_0^3 A\overline{A}^2 e^{-i\omega_0 T_0} + i\omega_0^3 \overline{A}^3 e^{-3i\omega_0 T_0} + \right] + \frac{1}{2} f\left[e^{i(\omega_0 T_0 + \alpha T_1)} + e^{-i(\omega_0 T_0 + \alpha T_1)}\right], \quad (31)$$

i.e.

$$\frac{\partial^2 x_1}{\partial T_0^2} + \omega_0^2 x_1 = -\left[2i\omega_0\left(\frac{\partial A}{\partial T_1} + \zeta_1 A\right) + 3i\omega_0^3 A^2 \overline{A}\right] e^{i\omega_0 T_0} + \zeta_p i\omega_0^3 A^3 e^{3i\omega_0 T_0} + \frac{1}{2} f e^{i(\omega_0 T_0 + \alpha T_1)} + cc, \quad (32)$$

where **cc** marks the complex conjugate of the previous terms.
But

$$e^{i(\omega_0 T_0 + \alpha T_1)} = e^{i\omega_0 T_0} e^{i\alpha T_1} \quad (33)$$

and therefore the secular terms will disappear in the solution of (32) if

$$2i\omega_0\left(\frac{\partial A}{\partial T_1} + \zeta_1 A\right) + \frac{1}{2} f e^{i\alpha T_1} = 0. \quad (34)$$

Let us consider

$$A = \frac{1}{2} a e^{i\beta}, \quad (35)$$

where a and β are real and functions of T_1.
We obtain

$$\frac{\partial A}{\partial T_1} = \frac{1}{2}\frac{\partial a}{\partial T_1} e^{i\beta} + \frac{1}{2} ai \frac{\partial \beta}{\partial T_1} e^{i\beta}, \quad (36)$$

$$A^2 = \frac{1}{4} a^2 e^{2i\beta}, \quad (37)$$

$$\overline{A} = \frac{1}{2} a e^{-i\beta}, \quad (38)$$

$$A^2 \overline{A} = \frac{1}{8} a^3 e^{i\beta} \quad (39)$$

and (34) becomes

$$2i\omega_0\left(\frac{1}{2}\frac{\partial a}{\partial T_1} e^{i\beta} + \frac{1}{2} ai \frac{\partial \beta}{\partial T_1} e^{i\beta}\right) + \frac{3i}{8}\omega_0^3 a^3 e^{i\beta} + \frac{1}{2} f e^{i\alpha_1} = 0, \quad (40)$$

that is,

$$2i\omega_0\left[\frac{1}{2}\frac{\partial a}{\partial T_1}(\cos\beta + i\sin\beta) + \frac{1}{2}ai\frac{\partial \beta}{\partial T_1}(\cos\beta + i\sin\beta)\right] + \frac{3i}{8}\omega_0^3 a^3(\cos\beta + i\sin\beta) + \frac{1}{2}f(\cos\alpha T_1 + i\sin\alpha T_1) = 0 \quad (41)$$

and separating the real and the imaginary parts, it results

$$-\omega_0\frac{\partial a}{\partial T_1}\sin\beta - a\omega_0\frac{\partial \beta}{\partial T_1}\cos\beta - \frac{3}{8}\omega_0^3 a^3\sin\beta + \frac{1}{2}f\cos\alpha T_1 = 0, \quad (42)$$

$$\omega_0\frac{\partial a}{\partial T_1}\cos\beta - \omega_0 a\frac{\partial \beta_1}{\partial T_1}\sin\beta + \frac{3}{8}\omega_0^3 a^3\cos\beta + \frac{1}{2}f\sin\alpha T_1 = 0. \quad (43)$$

Multiplying now (42) by $-\sin\beta$, (43) by $\cos\beta$ and summing, we obtain

$$-\omega_0\frac{\partial a}{\partial T_1} + \frac{3}{8}\omega_0^3 a^3 + \frac{1}{2}f\sin(\alpha T_1 - \beta) = 0, \quad (44)$$

i.e.

$$\frac{\partial a}{\partial T_1} = \frac{3}{8}\omega_0^2 a^3 + \frac{1}{2\omega_0}f\sin(\alpha T_1 - \beta). \quad (45)$$

Analogically, multiplying (42) by $\cos\beta$, (43) by $\sin\beta$ and summing, we get

$$-\omega_0 a\frac{\partial \beta}{\partial T_1} + \frac{1}{2}f\cos(\alpha T_1 - \beta) = 0, \quad (46)$$

wherefrom

$$a\frac{\partial \beta}{\partial T_1} = \frac{1}{2\omega_0}f\cos(\alpha T_1 - \beta). \quad (47)$$

Relations (45) and (47) form a system of two nonlinear ordinary differential equations. Solving this system, we obtain $a = a(T_1)$, $\beta = \beta(T_1)$ and the approximate solution

$$x(t) = a(T_1)\cos(\omega_0 T_0 + \beta(T_1)) + O(\varepsilon). \quad (48)$$

IV. Steady-State Motion

This type of motion is characterized by [3], [4], [5]

$$\frac{\partial a}{\partial T_1} = 0, \ \frac{\partial \beta}{\partial T_1} = 0. \quad (49)$$

Relations (45) and (47) lead to

$$\frac{3}{8}\omega_0^2 a^3 + \frac{1}{2\omega_0}f\sin(\alpha T_1 - \beta) = 0 \quad (50)$$

and

$$\frac{1}{2\omega_0}f\cos(\alpha T_1 - \beta) = 0, \quad (51)$$

respectively.

Squaring the last two relations and summing the results, we obtain

$$\frac{1}{4\omega_0^2}f^2 = \frac{9}{64}\omega_0^4 a^6, \quad (52)$$

wherefrom

$$a^6 = \frac{16f^2}{\omega_0^6}; \quad (53)$$

hence

$$a = \frac{\sqrt[3]{4f}}{\omega_0}. \quad (54)$$

From the second relation in (48) results

$$\cos(\alpha T_1 - \beta) = 0, \quad (55)$$

so that

$$\alpha T_1 - \beta = \frac{\pi}{2}, \ \beta = \alpha T_1 - \frac{\pi}{2} \quad (56)$$

and the approximate solution takes the form

$$x = a\cos(\omega t - \gamma), \quad (57)$$

in which

$$\gamma = \alpha T_1 - \beta = \frac{\pi}{2} \quad (58)$$

and therefore

$$\mathbf{x} = \mathbf{a} \sin \omega \mathbf{t} . \qquad (59)$$

V. NUMERICAL SIMULATION

For the numerical simulation, we will consider the following realistic values: $\mathbf{m} = \mathbf{1}\,\mathrm{kg}$, $\mathbf{k} = \mathbf{100}\,\mathrm{N/m}$, $\mathbf{F_0} = \mathbf{30}\,\mathrm{N}$, $\boldsymbol{\omega} = \mathbf{6}\,\mathrm{rad/s}$, $\mathbf{c_1} = \mathbf{0.1}\,\mathrm{Ns/m}$, $\mathbf{p} = \mathbf{3}$, $\mathbf{c_p} = \mathbf{0.01}\,\mathrm{Ns^3/m^3}$.

The equation (4) is integrated numerically, using the Runge–Kutta fourth order method [2]. Firstly, (4) is written in the form

$$\dot{\xi}_1 = \xi_2 ,\ \dot{\xi}_2 = -\frac{\mathbf{k}}{\mathbf{m}}\xi_1 - \frac{\mathbf{c_1}}{\mathbf{m}}\xi_2 - \frac{\mathbf{c_p}}{\mathbf{m}}\xi_2^{\mathbf{p}} + \frac{\mathbf{F_0}}{\mathbf{m}}\cos \omega \mathbf{t} , \qquad (60)$$

where

$$\xi_1 = \mathbf{x},\ \xi_2 = \dot{\mathbf{x}} , \qquad (61)$$

The integration step was chosen as $\boldsymbol{\Delta t} = \mathbf{0.001}\,\mathrm{s}$. We made 10000 steps of integration, the first 2000 steps being ignored.

The initial conditions are

$$\mathbf{x}(\mathbf{0}) = \mathbf{0.1}\,\mathrm{m},\ \dot{\mathbf{x}}(\mathbf{0}) = \mathbf{0}\,\mathrm{m/s} , \qquad (62)$$

The resulted values are plotted in the diagrams capture in Fig. 2. for $\mathbf{x}$, and in Fig. 3. for $\dot{\mathbf{x}}$, respectively.

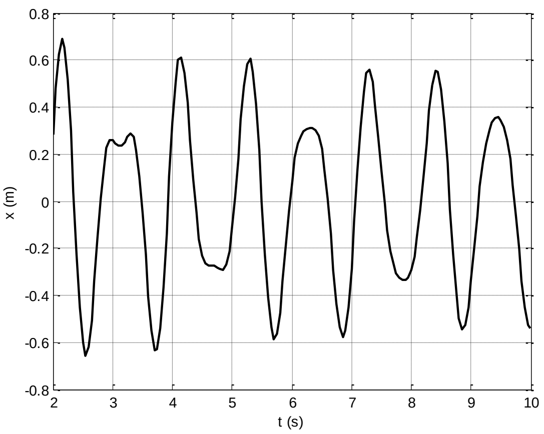

Fig. 2. Time history for the variable $\mathbf{x} = \mathbf{x}(\mathbf{t})$, obtained by numerical simulation and using (4).

The diagrams show a quasi-periodical motion, the quasi-period having values around $1\,\mathrm{rad/s}$. The amplitude of the motion has values between $0.25\,\mathrm{m}$ and $0.7\,\mathrm{m}$. The velocity of the mass is limited to $5\,\mathrm{m/s}$.

The diagrams plotted in Fig. 4 and Fig. 5 suggested that the point characterized by $\mathbf{x} = \mathbf{0}$ is a simply stable one.

Indeed, using the theory developed in [3], [4], the characteristic equation takes the form

$$\lambda^2 + \frac{\mathbf{c_1}}{\mathbf{m}}\lambda + \frac{\mathbf{k}}{\mathbf{m}} = \mathbf{0} \qquad (63)$$

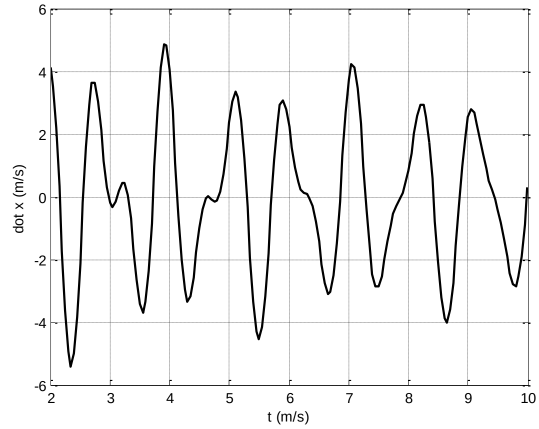

Fig. 3. Time history for the variable $\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{t})$, obtained by numerical simulation and using (4).

and it has no positive real roots, according to the Descartes theorem [2].

The condition of simple stable motion [3], [4], [5], is equivalent to the condition that (63) have two distinct roots, with negative real parts, i.e.

$$\frac{\mathbf{c_1^2}}{\mathbf{m^2}} - \mathbf{4}\frac{\mathbf{k}}{\mathbf{m}} < \mathbf{0} . \qquad (64)$$

This condition is obviously satisfied in our case.

In Fig. 4. we plotted the results obtained by numerical simulation for (48), with the parameters $\mathbf{a}$ and $\boldsymbol{\beta}$ given by (45) and (47). The reader can easily observe the good agreement between the diagrams plotted in Fig. 2. and Fig. 4., respectively.

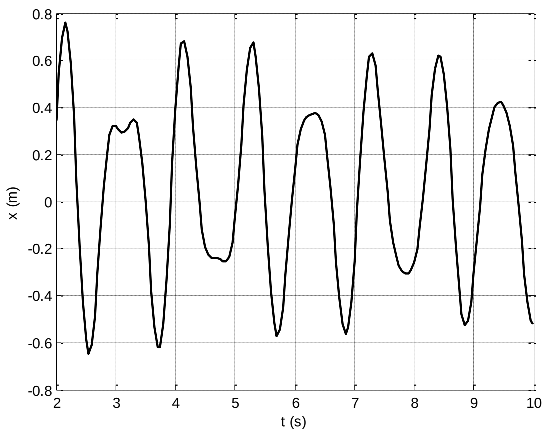

Fig. 4. Time history for the variable $\mathbf{x} = \mathbf{x}(\mathbf{t})$, obtained by numerical simulation and using (48).

The error of the method used in the paper and the real value for the variable $\mathbf{x} = \mathbf{x}(\mathbf{t})$ was calculated as the difference between the value of $\mathbf{x} = \mathbf{x}(\mathbf{t})$ given by (4) and the value $\mathbf{x} = \mathbf{x}(\mathbf{t})$ given by (48). The results thus obtained were plotted in Fig. 5., where the maximum error is about $0.01\,\mathrm{m}$. Keeping into account the values plotted in Fig. 2., we may conclude that the error is less than 5%.

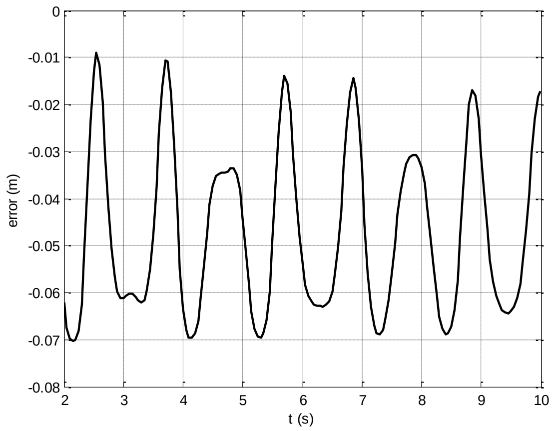

Fig. 5. The difference between the values calculated for $\mathbf{x} = \mathbf{x}(\mathbf{t})$ using (4) and (48).

Finally, in Fig. 6. we plotted the steady-state motion, using (59), with $\mathbf{a}$ given by (4). In this case, the steady-state motion is a harmonic motion of period

$$\mathbf{T} = \frac{2\pi}{\omega} \approx \mathbf{1.047}\,\mathrm{s}. \qquad (65)$$

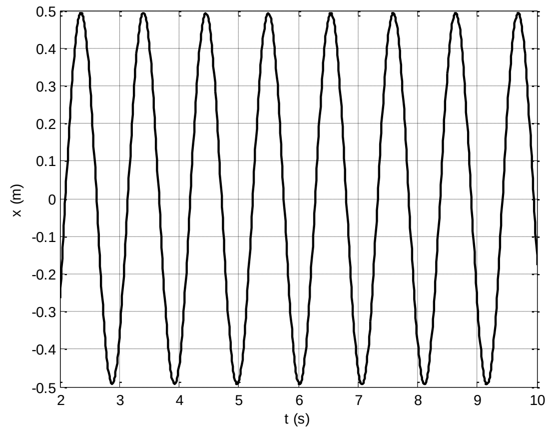

Fig. 6. The steady-state motion.

VI. CONCLUSIONS

In this paper we treated the forced vibrations with $\mathbf{p}$ th order damper for a single degree of freedom system. We consider that the elastic force is linear in displacement, and the damping force contains two terms one linear in velocity and the second as a power of $\mathbf{p}$ th order in the same velocity. The differential equation of motion was studied using multiple scale method and was also discussed the steady-state motion, this motion being a harmonic one with the pulsation equal to that of the exciting force.

The values obtained by the analytical method developed in the paper were compared with those obtained by numerical simulation, using directly the equation of motion. The agreement between the two sets of values is a very good one, the error being less than 5%. This indicates the accuracy of the analytical method used in the paper. If one wishes to increase the precision of the analytical method, one may use more terms in the development of $\mathbf{x}$; hence the parameters $\mathbf{a}$ and $\boldsymbol{\beta}$ are determined more précis, and the accuracy will increase.

The method can be used if and only if it is applied near a saddle point [1], [3], [4], [5]. The validation of the method is performed in the numerical case where the condition of saddle point was determined as (64) and verified for the numerical values considered in the application. Moreover, the condition (64) is also the condition for the stability of motion [3], [4], [5] and we conclude that in the numerical case not only the point $\mathbf{x} = \mathbf{0}$ is simply stable, but also the motions around it are simply stable. Let us observe that if the spring disappears, that is $\mathbf{k} = \mathbf{0}$, we obtain from (63) a solution $\boldsymbol{\lambda} = \mathbf{0}$, and a negative one, the stability remaining unchanged.

If the parameter $\mathbf{c_1}$ becomes positive, than (64) leads to positive or complex with positive real part, the system becoming an unstable one. The parameter $\mathbf{c_p}$ does not influence the stability. Its influence is reduced to the amplitude and period of the motion.

The method presented here can be used for any systems that can be brought to the form (4).

A more laborious calculation may be performed for any $\mathbf{p}$ positive, integer and greater than unity, the results being similar, with the same conclusions regarding the stability and shape of the diagrams.

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