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HOPF INVARIANT AND ITS ROLE IN HOMOTROPICS OF MAPPING SPHERE TO SPHERE

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***Abstract*—**Hopf invariant is introduced in order to evidence the existence of an endless set of classes of copying three-dimensional sphere in two-dimensional sphere. Later, Hopf has defined this invariant in order to copying (2k+1)-dimensional sphere in (k+1)-dimensional sphere.

***Keywords*—**Invariant, copying sphere, multiplicity,

I. INTRODUCTION

HAUSDORFF space and Hopf invariant have a significant place and role in the theory of homotopy and wider in topology. Hopf invariant is an invariant of homotopic class of mapping topological spaces. It was defined by Hopf for mapping the sphere $f:S^{2n-1} \rightarrow S^n$. Let's say that $f: S^{2n-1} \rightarrow S^n$ is continuous mapping.

When transferring on homotopic mapping, we can consider that mapping to be simplified in relation to some triangulations of the spheres S^n and S^{2n-1}. Then, Hopf invariant is defined as hanging coefficient of (n-1)-dimensional submultiplicities that intersect–$f^{-1}(a)$ and $f^{-1}(b)$ in S^{2n-1} for each different $a, b \in S^n$.

Definition of hanging coefficient: Let's say that M^k and N^l are two closed smooth oriented multiplicities of proportion k and l, and f and g are their continuous mappings in oriented Euclidean space E^{k+l+1} of proportion k+l+1, where the sets $f(M^k)$ and $g(N^l)$ do not intersect. Furthermore, let's say that S^{k+l} is a unit sphere of the space E^{k+l+1} with centre in an arbitrary point s O, taken with that orientation, which it has as a border of ball and $M^k \times N^l$ oriented right product of multiplicity of M^k and N^l. Each point $(x,y) \in M^k \times N^l$, $x \in M^k$, $y \in N^l$ corresponds to a non-zero segment (f(x), g(y)), and intersection of that ray with S^{k+1} we shall mark with X(x,y). Level of mapping X oriented multiplicity of $M^k \times N^l$ in oriented sphere S^{k+1} is called *hanging coefficient* of mapped multiplicities (f,M^k) and (g,N^l) and it is marked with $b((f,M^k), (g,N^l))$. Obviously, if mappings f and g change constantly: $f=f_t$, $g=g_t$ – in a way that sets $f_t(M^k)$ and $g_t(N^l)$ do not intersect in any t, then mapping $X=X_t$ is also changed constantly and, for that reason, hanging coefficient is not changed. Frequently, when multiplicities occur as submultiplicities of the space E^{k+l+1} and mappings f and g are identical, hanging coefficient is also determined and in that case it is marked with $b(M^k, N^l)$. Actually

$b((g, N^l), (f,M^k)) = (-1)^{(k+1)(l+1)} b((f, M^k), (g, N^l))$ (1)

This formula can be proven in the following manner: Let's say that χ' is the mapping of multiplicities $N^l \times M^k$ into S^{k+l}, analogous to previously derived mapping of χ. With the symbol λ, we shall mark the mapping of multiplicity $N^l \times M^k$ into multiplicity $M^k \times N^l$, which translates the point (y, x) into the point (x, y) and let's say that μ is mapping of the sphere S^{k+l} to itself, which translates each its point into a diametrically opposite. It is obvious that the level of mapping λ equals $(-1)^{k+l}$, and level of mapping μ equals $(-1)^{k+l+1}$. It can easily be seen that $\chi' = \mu \chi \lambda$. From this, the accuracy of above formula follows (1).

Now, let's say that instead of one mapped multiplicity (g, N^l), there are two multiplicities (g_0, N_0^l) and (g_1, N_1^l). Let's say that oriented limited component of multiplicity is still N^{l+1}, whose oriented limit consists of multiplicities $N_0^l$and $-N_1^l$, and there is mapping of g multiplicities of N^{l+1} in space E^{k+l+1}, which overlaps with g_0 on N_0^1 and with g_1 on N_1^1 where sets f (M^k) and $g(N^{l+1})$ do not mutually intersect. Actually

$b((f, M^k), (g_0, N_0^l)) = b((f, M^k), (g_1, N_1^l))$ (2)

Let's prove it. As a limit of multiplicity $M^k \times N^{l+1}$ we use multiplicities $M^k \times N_0^l$- $M^k \times N_1^l$. Each point $(x,y) \in M^k \times N^{l+1}$ corresponds with a segment (f(x), g(y)). Let's install a ray from point O, parallel with segment (f(x), g(y)) and mark its intersection with the sphere S^{k+1} as $\chi(x,y)$. Thus, we obtain continuous mapping χ of multiplicities $M^k \times N^{l+1}$ into the sphere S^{k+1} and, for that reason, the level of mapping of χ at its border equals zero. From this, the accuracy of formula (2) directly follows.

Mapping $f:S^{2n-1} \rightarrow S^n$ is determined by the element $[f] \in \pi_{2n-1}(S^n)$ and the image of the element $[f]$ in case of homeomorphism $\pi_{2n-1}(s^n) = \pi_{2n-2}(S^n) \xrightarrow{h} H_{2n-2}(\Omega S^n) = Z$ matches Hopf invariant H(f) (h is homeomorphism here).

Let's now take that $f: S^{2n-1} \rightarrow S^n$ is mapping of the class S^2

and form $\Omega\in\lambda^n S^n$ represents a group of integer homologies $H^n(S^n,Z)$. As such a form, we can take, for example, form $\Omega=\frac{dV}{vol(S^n)}$ where dV is the element of scope on S^n in some matrix and $vol(S^n)$ is the scope of sphere S^n. Then the form $f^x(\Omega)\in\lambda^n S^{2n-1}$ is closed and due to triviality of the group H^n, $H^n(S^{2n-1},Z)$ it appears as a point. Therefore, $f^x(\Omega)=d\theta$ for some forms $\theta\in\lambda^{n-1}S^{2n-1}$. There is a formula for calculation of Hopf invariant.
$H(f)=\int S^{2n-1}\theta\lambda\, d\theta$.

II. Pontryagin's interpretation of Hopf invariant

Definition: Let's say that f is smooth mapping of oriented sphere Σ^{2k+1} of the proportion 2k+1 into oriented sphere S^{k+1} of the proportion k+1, k≥1. Further, let's say that p' and q' are north and south poles of the sphere Σ^{2k+1}; E^{2k+1} is tangent line on the sphere Σ^{2k+1} in the point p' and φ central projection of sets $\Sigma^{2k+1}\backslash q'$ on the space E^{2k+1}. In the sphere S^{k+1} we shall select two different among each other and from f(q') regular points A_0 and A_1 of the mapping f; then $M_0^k=f^{-1}(a_0)$ and $M_1^k=f^{-1}(a_1)$ are closed oriented submultiplicities of Euclidean space E^{2k+1}. Let's say that $\gamma(f)=\gamma(f,p',a_0,a_1)=b(M_0^k, M_1^k)$. Actually, γ(f) is a homotopic invariant of the mapping f that does not depend on random selection of points p', a_0, a_1 and that for numerical k that invariant is always equal to zero.
We can show the invariance of the number γ(f).

Let's say that f_0 and f_1 are two smooth homotoph mappings of the sphere Σ^{2k+1} in S^{k+1} and f_t smooth deformation that connects them. Deformation f_t coincides with mapping of f_x product of $\Sigma^{2k+1}\times I$ into S^{k+1}. We can observe that with sufficiently small movement of points a_0 and a_1 the number $\gamma(f_t)$, t=0,1 is not changed since multiplicities $\varphi f_t^{-1}(a_i)$ suffer only s small deformation. For that reason, we can observe that the curve $f_t(q')$, 0≤t≤1, does not go through the points a_o and a_1. Let's say that r is so big natural number that for $|t'-t|<\frac{1}{r}$ the sets $f_t^{-1}(a_0)$ and $f_t^{-1}(a_1)$ do not mutually intersect. Now, let's move away from the points a_0 and a_1 so that they become proper mapping points f_x and f_t; t=0,$\frac{1}{r}$, … , $\frac{r-1}{r}$, 1.

Let's now prove that $\gamma(f_1)=\gamma(f_0)$.

Part of the segment I that consists of the points, which satisfy the condition $\frac{s}{r}\le t\le\frac{s+1}{r}$, we will mark with I_s, and let's say that $M_{s,i}^{k+1}$ is the original of the point a_i in the lane $M^k\times I_s$ in case of mapping of the f_x. Due to conditions set for points $a_0<a_1$, the set $M_{s,i}^{k+1}$ is oriented submultiplicity of multiplicity $\Sigma^{2k+1}\times I_s$, whose border is multiplicity $f_{\frac{s}{r}}^{-1}(a_i)+f_{\frac{s+1}{r}}^{-1}(a_i)$. Projection of multiplicity $\Sigma^{2k+1}\times I$ along the axis I onto the sphere Σ^{2k+1} we will mark with π. Mapping of φπ of multiplicity $M_{s,i}^{k+1}$ is determined by mapping the multiplicity $(\varphi\pi, M_{s,i}^{k+1})$ with a border $\varphi f_{\frac{s}{r}}^{-1}(a_i)+\varphi f_{\frac{s+1}{r}}^{-1}(a_i)$. Since the sets $\varphi\pi(M_{s,0}^{k+1})$ and $\varphi\pi(M_{s,1}^{k+1})$ do not mutually intersect, it follows that $\gamma(f_{\frac{s+1}{r}})=\gamma(f_{\frac{s}{r}})$, and for that reason $\gamma(f_1)=\gamma(f_0)$.

Now, let's prove that $\gamma(f, p', a_0, a_1)$ does not depend on the selection of points a_0 and a_1. Let's say that instead of the points a_0 and a_1, the points b_0 and b_1 were selected. Obviously, there is a smooth homeomorphism λ of the sphere S^{k+1} to itself, homotopic to identical, where $\lambda(a_i)=b_i$, i=0,1. It is obvious that $\gamma(\lambda f, p', b_0, b_1)=\gamma(f, p', a_0, a_1)$, and since mappings of λf and f are homotopic among themselves, we obtain $\gamma(f, p', b_0, b_1)=\gamma(f, p', a_0, a_1)$.

Analogous, it is proved that $\gamma(f, p', a_0, a_1)$ does not depend on the selection of the point p', since there is a rotation of the sphere Σ^{2k+1}, which translates the point p' into another arbitrary point of the sphere Σ^{2k+1}.

In the end, let's show that, due to even k the invariant γ(f) is transformed to 0. Since the number γ(f) does not depend on the selection of points p_0 and f_1, then we can change their roles and then we have $b(M_0^k, M_1^k)=b(M_1^k, M_0^k)$. Since $b(M_1^k, M_0^k)=(-1)^{(k+1)^2}b(M_0^k, M_1^k)$, then in case of even k, we have $b(M_0^k, M_1^k)=0$.

III. Hopf invariant of equipped multiplicity

If homotopic classes of the mapping of (2k+1)-dimensional sphere into (k+1)-dimensional sphere are in mutually unambiguous correspondence with homologous classes of equipped k-dimensional multiplicities of (2k+1)-dimensional space, then invariant γ(f) can be expressed as invariant of homological class of equipped k-dimensional multiplicities in (2k+1)-dimensional space.
Now, let's explain the invariant γ(f).

1) Let's say that (M^k, U), $U(x)=\{U_1(x),\ldots,U_{k+i}(x)\}$, is equipped submultiplicity of oriented Euclidean space E^{2k+1} and N_x is normal to multiplicity on M^k in the point $x\in M^k$. That normal will be observed as vector space with the beginning in point x so that U(x) is the base of the space N_x. Let's arbitrary select a vector $c=\{c^1,\ldots,c^{k+1}\}$ of some coordinate Euclidean space N and let's say that point $x\in M^k$ corresponds with the point $c(x)=c^1U_1(x)+\ldots+c^{k+1}U_{k+1}(x)$ of the space N_x. In case of sufficiently small vector c, the mapping of C is homeomorphous mapping of multiplicity M^k into the space E^{2k+1}. It is obvious that if c≠0 multiplicities M^k and $c(M^k)$ do not intersect among themselves and that for two different non-zero vectors c and c' multiplicities $c(M^k)$ and $c'(M^k)$ are homotopic among themselves in the space $E^{2k+1}\backslash M^k$. According to this, for a sufficiently small, different than zero, vector c, hanging coefficient $b(M^k, c(M^k))$ does not depend on vector c and we shall put
$\gamma(M^k,U)=b(M^k, c(M^k))$.

Actually, if $f \to (M^k, U)$ then $\gamma(f)=\gamma(M^k,U)$.
Since $\gamma(f)$ is a homotopic invariant of the mapping f, then $\gamma(M^k,U)$ is homotopic invariant of equipped multiplicity (M^k,U).
Formula $\gamma(f)=\gamma(M^k,U)$ can be proven in the following way. Let's say that f is smooth mapping of the sphere $\sum^{2k+1}$ into the sphere S^{k+1} and $p \in S^{k+1}$ is the proper point of mapping the f, different than f(q'). Then, for the construction of equipped multiplicity (M^k,U), which corresponds to the mapping of f, the point p can be taken as north pole of the sphere S^{2k+1}. Let's say that $e_1, \dots, e_{k+1}$ is some orthogonal base of tangent plane in the point p onto the sphere S^{k+1} and $x^1, \dots, x^{k+1}$ of the appropriate coordinate of that base in the area $S^{k+1}\backslash$-q. For the construction of invariant $\gamma(f)$ we take the point a_0 pol p, and for the point a_1 point with coordinates $x^1=c^1, \dots, x^{k+1}=c^{k+1}$. In such selection of points a_0 and a_1 multiplicity M_1^k obviously corresponds with multiplicity M^k, and multiplicity M_1^k is placed from multiplicity $c(M^k)$ depending on the size of vector c. For that reason, $b(M^k, C(M^k))= b(M_0^k, M_1^k)$ and relation $\gamma(f)=\gamma(M^k,U)$ is proven.

2) Let's say that $\prod_{k+1}^k$ is a homologous group of equipped k-dimensional multiplicities of Euclidean oriented space E^{2k+1}. Each element of $\pi \in \prod_{k+1}^k$ corresponds with integer $\gamma(\pi)=\gamma(M^k,U)$ where (M^k,U) is equipped multiplicity of the class π. The number $\gamma(\pi)$ depends only on the element π and it does not depend on random choice of equipped multiplicity (M^k,U). Actually, γ is homomorphous mapping of the group $\prod_{k+1}^k$ in additive group of integers. From this, it follows that a set $\prod_{k+1}^k$ of all elements of $\pi \in \prod_{k+1}^k$, for which $\gamma(\pi)=0$, is a subset of $\prod_{k+1}^k$.

It can proven in the following manner: Let's say that π_1 and π_2 are two elements of the group $\prod_{k+1}^k$, and (M_1^k,U_1) and (M_2^k,U_2) are equipped multiplicities that correspond to classes π_1 and π_2 and which lie on different sides from some hyperplane E^{2k} of the space E^{2k+1}. Further, let's say that S^{2k} is the unit sphere of the space E^{2k+1} with a centre O, which belongs to E^{2k}. Let's arbitrarily select a small vector c that determines the replacement of multiplicity $M_1^k \cup M_2^k$. We have
$\gamma(\pi_1+\pi_2)=b(M_1^k \cup M_2^k, c(M_1^k \cup M_2^k))$.
Hanging coefficient, which is found on the right side, is defined as a level of mapping X of the multiplicity $(M_1^k \cup M_2^k)\times c(M_1^k \cup M_2^k)$ onto the sphere S^{2k}. Level of mapping of X will be determined in the point p of the sphere S^{2k}, which lies close to the hyperplane E^{2k}. Owing to such selection of the point p segment (x, C(y)), where $x \in M_1^k$, $y \in M_2^k$, cannot be parallel with the segment (0,p). From this it follows that

$$b(M_1^k \cup M_2^k, c(M_1^k \cup M_2^k))= b(M_1^k, c(M_1^k))+ b(M_2^k, c(M_2^k)),$$

i.e. further

$$\gamma(\pi_1+\pi_2)= \gamma(\pi_1)+ \gamma(\pi_2).$$

According to this, the attitude is proved.

3) Let's say that f is smooth mapping of oriented sphere $\sum^{2k+1}$ onto the oriented sphere S^{2k+1}, g mapping of the sphere $\sum^{2k+1}$ to itself with the degree σ, and h mapping of the sphere S^{2k+1} to itself with the degree τ. We will put that f'=h f g. It appears that

$$\gamma(f')= \sigma \tau^2 \gamma(f).$$

This attitude is enough to prove for the case when h is identical mapping and for the case when g is identical mapping. We must consider the case when g is identical mapping, i.e. when f'=hf. Let's say that a_0 and a_1 are two different than f'(q'), points of the sphere S^{k+1}, which appear as proper points of mapping h and hf. Then $h^{-1}(a_t)=\{a_{t1}, \dots, a_{tr_t}\}$; t=0,1, where mapping of f is proper in any of the points a_{t_i}, t=0,1, i=1,2, … , r_t. The sign of functional determinant of mapping of h in the point a_{t_i} we shall mark with ε_{t_i}, i=1,2, … , r_t, t=0,1. Tangent in the north pole p' of the sphere $\sum^{2k+1}$ we will mark with E^{2k+1}, and central mapping of the set $\sum^{2k+1}\backslash q'$ on it from the point q' from φ. Let's put that $\varphi f'^{-1}(a_t)=M_t^k$, t=0,1; $\varphi f^{-1}(a_{t_i})=M_{t_i}^k$. It is easy to observe that

$$M_t^k=\varepsilon_{t_1} M_{t_1}^k \cup \varepsilon_{t_2} M_{t_2}^k \cup \dots \cup \varepsilon_{tr_t} M_{tr_t}^k$$

where the signs ε_{t_i} depend on orientation of the original. Since a_{0_i} and a_{1_j} are two different points of the sphere S^{k+1}, which appear as proper points of mapping f, then invariant (f) can be defined as $b(M_{0_i}^k, M_{1_j}^k)$. From this it follows that

$$\gamma(f)=b(\varepsilon_{0_1} M_{0_1}^k \cup \dots \cup \varepsilon_{0r_0} M_{0r_0}^k \cup \varepsilon_{1_1} M_{1_1}^k \cup \dots \cup \varepsilon_{1r_1} M_{1r_1}^k)=$$

$$\sum_{i=1}^{r_0} \sum_{j=1}^{r_1} \varepsilon_{0_i} \varepsilon_{1_j} \gamma(f) = \gamma(f)(\sum_{i=1}^{r_0} \varepsilon_{0_i})(\sum_{j=1}^{r_1} \varepsilon_{1_j}) = \tau^2 \gamma(f).$$

Thus, this attitude is proved.

IV. Conclusion

Hopf invariant has an important role in homotopic classification of mapping of sphere into sphere and it was initially introduced in order to prove the existence of uncountable set of classes of mapping the three-dimensional sphere to two-dimensional, and then for the sake of mapping (2k+1)-dimensional sphere into (k+1)-dimensional. Hopf invariant is defined and, in the end, described with the help of equipped multiplicity that corresponds with mapping.

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