ANNALS OF THE ORADEA UNIVERSITY
Fascicle of Management and Technological Engineering
ISSUE #2, SEPTEMBER 2013, http://www.imtuoradea.ro/auo.fmte/

ONE-AMPLITUDINAL AND TWO-AMPLITUDINAL SOLUTIONS OF THE CANONICAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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***Abstract*—**In this work, we have determined one-amplitudinal and two-amplitudinal solutions of the canonical differential equation of the second order (1.1.), in case when it is oscillating. The canonical differential equation of the second order is oscillating when the coefficient of the equation is a positive, uninterrupted function which is sufficiently big to cause oscillations. We have also determined the function of frequency g(x) and shown that it is linear function of x. Finally, we have determined conditions under which the equation (1.1) is Hill's equation and has two-particular, two-amplitudinal periodic solutions. The idea can be rather interesting when talking about the differential equations with singularities, the solutions of which are special functions: Bessel function, hyper-geometric functions, as well as Legendre, Hermite and Laguerre's polynomials**.**

***Keywords*—**Differential equations, one-amplitudinal and two- amplitudinal solutions, method of iteration sequences, function of frequency, generalized sine and cosine.

I. INTRODUCTION

IT is familiar that the canonical equation

$$y'' + a(x)\cdot y = 0 \qquad (1.1.)$$

can be solved by applying the method of iteration sequences if a(x) is positive uninterrupted function and if the integral $\int_0^{+\infty} a(x)dx$ diverges. By applying this method, two linearly independent particular solutions $y_1$and y_2 are obtained. These have forms of uniformly convergent sequences made up of multiple integrals, which we have justifiably called general (generalized) sine and cosine with the base $a(x)$. These solutions have the following forms:

$$\begin{cases} y_1 = \cos_{a(x)} x = 1 + \sum_{k=0}^{\infty} (-1)^k \int_0^x\int_0^x a(x)dx^2 \int_0^x\int_0^x a(x)dx^2 \ldots \int_0^x\int_0^x a(x)dx^2 \\ y_2 = \sin_{a(x)} x = x + \sum_{k=1}^{\infty} (-1)^k \int_0^x\int_0^x a(x)dx^2 \int_0^x\int_0^x a(x)dx^2 \ldots \int_0^x\int_0^x xa(x)dx^2 \end{cases}$$

(1.2.)

The very iterations do not show that the solutions (1.2.) are oscillating. The ocillatory nature has been proved indirectly by using the condition $a(x)>0$. We have also determined very precise approximate formulae for solutions $y_1$and y_2. These formulae are expressed with Euclidean sine and cosine of the complex function $x\sqrt{a(x)}$, i.e. it has been demonstrated that:

$$\begin{cases} y_1 = \cos_{a(x)} x \approx \cos(x\sqrt{a(x)}), \\ y_2 = \sin_{a(x)} x \approx \dfrac{1}{\sqrt{a(x)}} \sin(x\sqrt{a(x)}). \end{cases} \qquad (1.3.)$$

The Sturm theorems on closer locations of zeros of the oscillations are easily obtained based on the relation (1.3.). The zeros of the oscillations $y_1$and y_2 are approximately located in the solutions of the equation sequences:

$$\begin{cases} for \quad y_1 : x\sqrt{a(x)} = (2k-1)\dfrac{\pi}{2}, \quad k = 1,2,3,\ldots, \\ for \quad y_2 : x\sqrt{a(x)} = k\pi, \quad k = 0,1,2,\ldots \end{cases} \qquad (1.4.)$$

This provides rather easy and elementary approach to the problem of the locations of zeros, which completely differs from the existing approaches, that have been applicable for more than two centuries, i.e. since Sturm's time. However, for studying oscillations, apart from the number of zeros and their locations, there are also other important elements. These are primarily amplitudes, which are constant in harmonious oscillations. In other oscillations, which are not less important and, surely, are more frequent (there is no ideal processes during longer period of time), there are functions of the basic argument (distance *x* and time *t*). Since the solutions $y_1$and y_2 are approximated by relations (1.3.), there is:

Theorem 1.1. The solutions $y_1$and y_2 of the equation (1.1.) are two-amplitudinal and their amplitudes are as follows:

$$\begin{cases} A_1(\cos_{a(x)} x) \approx 1 \\ A_2(\sin_{a(x)} x) \approx \dfrac{1}{\sqrt{a(x)}} \end{cases} \qquad (1.5.)$$

Therefore, $|A_1| \le 1$, $|A_2| \to 0$, $x \to +\infty$, which again shows big advantages of the method of iteration sequences in qualitative analysis.

Theorem 1.2. The solutions $\begin{cases} y_1 = \cos_{a(x)} x = C(x), \\ y_2 = \sin_{a(x)} x = S(x), \end{cases}$ are completely different and there is no phase concurrence.

Theorem 1.3. The solutions $y_1$and y_2 of the equation (1.1.) are oscillating when $x \to +\infty$. However the solution y_1 oscillates, but remains limited. On the other hand, the solution y_2 tends to zero.

Therefore, the solutions $y_1$and y_2 meet the requirements of the Prodi's theorem.

In this work, the researches [6-8, 10-11, 14] continue in order to improve method of finding the amplitudes by relaying on formulae (1.2.), (1.3.), (1.4.) and (1.5.), as well as on analogies with some results with quadratures (see Berkovich, [1, pp. 65, 92, 99 and 107].

II. MAIN RESULTS

Some classes of the equations of oscillations (1.1.), primarily, when the coefficient of the equation is infinitely small function $a(x) = \dfrac{1}{x^\alpha}$, $0 < \alpha < 2$, can be solved quadraturely. Their solutions are one-amplitudinal and have the following form:

$$\begin{cases} y_1 = \varphi(x)\cos x, \\ y_2 = \varphi(x)\sin x. \end{cases} \qquad (2.1.)$$

Here $A(x) = \varphi(x)$ is variable amplitude, while *g(x)* is a so called function of frequency. If the functions φ and g are uninterrupted and twice uninterruptedly differentiable, then the equation with solutions (2.1.), according Liouville's formula is the following:

$$\begin{vmatrix} y'' & y' & y \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix} = 0 \qquad (2.2.)$$

Or in the developed form:

$$W(x)y'' + A(x)y' + B(x)y = 0 \qquad (2.3.)$$

where *W(x)*, *A(x)* and *B(x)* are minors of the determinant (2.2.).

Since

$$\begin{cases} y_1' = \varphi' \cos g - \varphi g' \sin g, \\ y_2' = \varphi' \sin g + \varphi g' \cos g, \end{cases} \qquad (2.4.)$$

then the Wronskian *W(x)* is given with:

$$W(x) = \begin{vmatrix} y_1' & y_1 \\ y_2' & y_2 \end{vmatrix} = y_1' y_2 - y_1 y_2' = -\varphi^2 \cdot g' \quad (2.5.)$$

Analogously, by using second derivatives:

$$\begin{cases} y_1'' = (\varphi'' - \varphi g'^2)\cos g - (\varphi g'' + 2\varphi' g')\sin g, \\ y_2'' = (\varphi g'' + 2\varphi' g')\cos g + (\varphi'' - \varphi g'^2)\sin g \end{cases} \qquad (2.6.)$$

the following is obtained:

$$A(x) = y_1 y_2'' - y_1'' y_2 = \varphi^2 g'' + 2\varphi\varphi' g', \qquad (2.7.)$$

$$B(x) = y_1'' y_2' - y_1' y_2'' = \left[\varphi g'(\varphi'' - \varphi g'^2) - \varphi g'(\varphi g'' + 2\varphi' g')\right] \cdot \cos^2 g -$$
$$- \left[\varphi'(\varphi g'' + 2\varphi' g') + \varphi'(\varphi'' - \varphi g'^2)\right] \cdot \sin^2 g + \left[(\varphi'' - \varphi g'^2)(\varphi' - \varphi g') - (\varphi g'' + 2\varphi' g')(\varphi' + \varphi g')\sin g \cos g\right]$$
(2.8.)

If $W(x) \ne 0$, the equation (2.3.), after being divided by *W(x)* is reduced to the following equation:

$$y'' + \frac{A(x)}{W(x)} y' + \frac{B(x)}{W(x)} y = 0 \qquad (2.9.)$$

Therefore the complete linear homogenous differential equation of the second order is obtained. It has solutions (2.1.) and in the same time it is:

$$y'' + a(x)y' + b(x)y = 0 \qquad (2.10.)$$

Finally, since it is necessary to obtain the canonical equation (1.1.), we shall take that $a(x) = \dfrac{A(x)}{W(x)} = 0$ in (2.10.). Since it is $a(x) = \dfrac{\frac{dW}{dx}}{W}$ based on Liouville's formula, then $\dfrac{dW}{dx} = 0$. Accordingly, $W(x) = const.$. Without reducing the generalization, we shall take that

$W(x) = -1$. Consequently, we obtain that the condition $A^2(x) \cdot g'(\alpha) = 1$ is valid for the variable amplitude, i.e.

$$A(x) = \frac{1}{\sqrt{g'(x)}} \quad (2.11.)$$

This is in complete accordance with the basic physical meaning that the amplitude is inversely proportional to the speed of the period change, i.e. frequency. Therefore, if particular solutions are (2.1.), the general solution is given in quadratures:

$$y(x) = \varphi(x)\left[C_1 \cos \int \frac{dx}{\varphi^2(x)} + C_2 \sin \int \frac{dx}{\varphi^2(x)}\right] \quad (2.12.)$$

The same is obtained by applying the function of frequency *g(x)*:

$$y(x) = \frac{1}{\sqrt{g'(x)}} \cdot [C_1 \cos g(x) + C_2 \sin g(x)] \quad (2.13.)$$

If solutions (2.1.) are one-amplitudinal solutions of the equation

$$y'' + b(x)y = 0 \quad (2.14.)$$

then the amplitude $A(x) = \varphi(x)$ can be determined in another manner. Truly, if solutions (2.1.) are solutions of the equation (2.14.), then the functions y_1 and y_2 identically fulfills the equation. Accordingly, the following identities are valid:

$$(\varphi'' - \varphi'^2)\cos g - (\varphi g'' + 2\varphi' g')\sin g + b(x)\varphi \cos g \equiv 0,$$

$$(\varphi g'' - 2\varphi' g')\cos g + (\varphi'' - \varphi g'^2)\sin g + b(x)\varphi \sin g \equiv 0,$$

i.e.

$$\begin{cases} \varphi'' \cos g - 2\varphi' g' \sin g - \varphi(g'' \sin g + g'^2 \cos g - b(x)\cos g) \equiv 0 \\ \varphi'' \sin g + 2\varphi' g' \cos g - \varphi(-g'' \cos g + g'^2 \sin g - b(x)\sin g) \equiv 0 \end{cases} \quad (2.15.)$$

If the first equation of the system (2.15.) is multiplied with cos*g* and if the second equation is multiplied with sin*g* and then those multiplied equations are summed, the canonical differential equation is obtained.

$$\varphi'' + (b(x) - g'^2) \cdot \varphi = 0 \quad (2.16)$$

The obtained equation is the (1.1.) equation type. Furthermore, it is the equation of linear oscillations if

1° $b(x) - g'^2(x) > 0$

2° the function $(b(x) - g'^2(x))\varphi$ fulfills the Lipschitz's condition.

Definition 2.1. The equation (2.16.) is the equation of amplitudes for the differential equation (1.1.).

The solution type of the equation (2.16.) depends on the coefficient $\phi(x) = b(x) - g'^2$. If the conditions 1° and 2° are fulfilled, the solutions are oscillating.

If $b(x) - g'^2 < 0$, then the equation (2.16.) is written in the following form:

$$\varphi'' = \left|b(x) - g'^2\right| \cdot \varphi \quad (2.17.)$$

By applying the method of iteration sequences on its equivalent integral equation:

$$\varphi = C_1 x + C_2 + \int_0^x \int_0^x \left|b(x) - g'^2(x)\right| \cdot \varphi(x) dx^2 , \quad (2.18.)$$

we obtain monotonous solutions which have maximum one zero, if there is zero at all according to Sturm's theorems. Finally, if $b(x) - g'^2 = 0$, we obtain that amplitude $A(x) = \varphi(x)$ is linear function $\varphi(x) = C_1 x + C_2$, while the function of frequency is given with $g(x) = \int \sqrt{b(x)} dx$.

Since the equation (1.1.), in respect with (1.3.), has two-amplitudinal solutions more frequently than one-amplitudinal (harmonious oscillations, as well as some others), there is the question: Under which conditions does the equation (1.1.) have two-amplitudinal solutions?

$$\begin{cases} y_1 = F(x)\cos g(x) \\ y_2 = G(x)\sin g(x), \end{cases} \quad (2.19.)$$

Here *F*, *G* and *g* are random frequencies, while *F* and *G* are also positive. In order to answer the above question, we shall follow the opposite direction. Namely, we shall form differential equation by using its two-amplitudinal solutions. Based on Liouville's formula, the required equation has the following developed form:

$$W(x)y'' + A(x)y' + B(x)y = 0 \quad (2.20.)$$

Whether the equation (2.20.) exists or not, it depends on the Wronskian *W(x)* i.e. if it is different from zero or it is equal to zero for every *x* from the observed interval.

By using the first derivatives

$$\begin{cases} y_1' = F'\cos g - Fg'\sin g \\ y_2' = G'\sin g + Gg'\cos g, \end{cases} \quad (2.21.)$$

we obtain:

$$W(x) = W(y_1 y_2) = (F'G - FG')\sin g\cos g - FGg' \quad (2.22.)$$

If *W(x)=0*, the following equation is obtained:

$$(F'G - FG')\sin g\cos g - FGg' = 0 \quad (2.23.)$$

The equation (2.23) is called resolvent equation of the required differential equation. It is also differential equation of the first order, but it is also linear and transcendent. If it is divided by F^2, the following is obtained:

$$\frac{F'G - FG'}{F^2} = \frac{F}{G}\cdot\frac{g'}{\sin g\cos g}, \text{ or}$$

$$\frac{d}{dx}\left(\frac{F}{G}\right) = \frac{F}{G}\cdot\frac{1}{\cos g\sin g}\cdot\frac{dg}{dx}, \text{ i.e.}$$

$$\frac{\frac{d}{dx}\left(\frac{F}{G}\right)}{\frac{F}{G}} = \frac{dg}{\sin g\cos g}$$

If the last equality is integralled, the following equation is obtained:

$$\frac{F(x)}{G(x)} = C\cdot tg\ g(x) \quad (2.24.)$$

which is, for the easier control of zeros, written in the following form:

$$F(x)\cdot\cos g(x) = C\cdot G(x)\cdot\sin g(x) \quad (2.25.)$$

The equation (2.25.) is identically fulfilled in the following cases:

1° When zeros of the function *F* are simultaneously zeros of the function *G*, i.e. when the equations *F(x)=0* and *G(x)=0* have common zeros.

2° When zeros of the function *F* are simultaneously zeros of the function sin*g(x)*, i.e. when the solutions of the equation $g(x) = n\pi$, *n=0,1,2,…* are zeros of the function *F*.

3° When zeros of the function cos*g(x)* are simultaneously zeros of the function *G*, i.e. when the solutions of the equation $g(x) = (2n-1)\frac{\pi}{2}$, *n=1,2,…* are zeros of the function *G*.

4° When zeros of the function cos*g(x)* are simultaneously zeros of the function sin*g(x)*, i.e. when $g(x) = (2n-1)\frac{\pi}{2}$, *n=1,2,…*is simultaneously $g(x) = n\pi$, *n=0,1,2,…*, which is not possible.

Therefore, the equation (2.23.) has solutions which identically satisfy it, if conditions 1°, 2° and 3° are fulfilled.

These conditions are not valid if $W(x) \neq 0$. It means that, in order to have normal linear homogenous differential equation of the second order without singularities, the amplitudes *F(x)* and *G(x)* cannot have zeros. This leads to:

Theorem 2.2. There is no linear homogenous differential equation of the second order with whole coefficients which has two-amplitudinal fundamental solutions (2.19.) if amplitudes *F* and *G* have zeros.

Therefore, in order to have uninterrupted linear oscillations of the second order, the amplitudes *F* and *G* must be without zeros, while zeros of the oscillations are only zeros of the functions cos*g(x)* and sin*g(x)*.

Theorem 2.3. There is linear homogenous differential equation of the second order which has exact (quadrature) solutions (2.19.) for uninterrupted functions *F* and *G* and condition $FG' - F'G \neq 0$

The theorem 2.3. is generalization of the classic Liouville-Besge theorem for exact quadrature solutions of the two-amplitudinal oscillations (1.1.). Here, this type of solution can be further developed, similarly as Berkovich did for one-amplitudinal solutions.

Now it is of high importance to determine the equation with the exact solutions (2.19.), for which $W(x) \neq 0$, i.e. for which conditions 1°, 2° and 3° are not valid. Since the required equation is as follows:

$$W(x)y'' + A(x)y' + B(x)y = 0,\quad W(x) \neq 0 \quad (2.26.)$$

the coefficients of the equation should be determined. Since the Wronskian has already been determined and the following has been obtained:

$$W(x) = (F'G - FG')\cos g\sin g - FGg' \quad (2.27.)$$

A(x) and *B(x)* should be also determined. As it is:

$$\begin{cases} y_1'' = (F'' - Fg'^2)\cos g - (2F'g' + FG'')\sin g \\ y_2'' = (G'' - Gg'^2)\sin g + (2G'g' + Gg'')\cos g, \end{cases} \quad (2.28.)$$

after some more calculation, the following is obtained:

$$\begin{cases} A(x) = FGg'' + 2F'Gg'\sin^2 g + 2FG'g'\cos^2 g + (FG'' - F''G)\sin g\cos g, \\ B(x) = \left[(G'' - Gg'^2)Fg' - (2F'g' + Fg'')G'\right]\sin^2 g + \end{cases} \quad (2.29.)$$

$$+\left[(F''-F'^2)Gg'-(2G'g'+Gg'')F'\right]\cos^2 g+$$

$$+\left[(F''-Fg'^2)G'-(2F'g'+Fg'')Gg'-(G''-Gg'^2)F'+(2G'g'+Gg'')Fg'\right]\sin g\cos g$$

Since we want to obtain the canonical equation:

$$y''+b(x)\cdot y=0\,,\quad b(x)=\frac{B(x)}{W(x)}\,, \qquad (2.30.)$$

we shall take that *W(x)=1* and *A(x)=0* and obtain the following equation:

$$y''+\left\{[(G''-Gg'^2)Fg'-(2F'g'+Fg'')G']\sin^2 g+[(F''-Fg'^2)Gg'-(2G'g'+Gg'')F']\cos^2 g+\right.$$
$$\left.+\left[(F''-Fg'^2)G'-(2F'g'+Fg'')Gg'-(G''-Gg'^2)F'+(2G'g'+G'')Fg'\right]\sin g\cos g\right\}\cdot y=0$$

(2.31.)

The equation (2.31.) seems complicated, but it is symmetrical. It can be considerably simplified. However, it would be good to use the fact that the functions cos*g(x)* and sin*g(x)* are periodic, because, in that case, the function of frequency *g(x)* can be more closely determined.

Let it be $\omega\in R$, $\omega>0$, of the period of the function cos*g(x)*. Then, regarding the definition of the periodicity, the following is valid:

$$\cos g(x+\omega)=\cos g(x) \qquad (2.32.)$$

Based on the definition of the elementary function cos*x*, there follows:

$$g(x+\omega)=g(x)+2k\pi\,,\ k=0,1,2,\ldots, \qquad (2.33.)$$

i.e.

$$g(x+\omega)-g(x)=2k\pi \qquad (2.34.)$$

If Lagrange's theorem on mean value is applied on the interval $[x,x+\omega]$, the following is obtained:

$$g'(\xi)[(x+\omega)-x]=2k\pi \ \text{ or}$$
$$g'(\xi)\cdot\omega=2k\pi \qquad (2.35.)$$

The number ω, which is obtained for *k=1* is called minimal period of the function cos*g(x)* and for *k=1*, the following is valid:

$$g(\xi)=\frac{2\pi}{\omega}=\alpha=const. \qquad (2.36.)$$

Since *x* is independently variable and it can change and increase, then ξ is not fixed, but it changes together with *x*. This leads to:

$$g'(\xi)=\alpha\Rightarrow\frac{dg(\xi)}{d\xi}=\alpha\Rightarrow dg(\xi)=\alpha\cdot d\xi\Rightarrow g(\xi)=\alpha\cdot\xi$$

Therefore, the function *g(x)* is linear function in the form of $g(x)=\alpha x$.

Theorem 2.4. The functions cos*g(x)* and sin*g(x)* are periodic if and only if *g(x)* is linear function:

$$g(x)=\alpha\cdot x. \qquad (2.37.)$$

Furthermore, if the functions *F* and *G* are periodic with the period 2π in the equation (2.29.), then the coefficient of the equation is periodic function. It means that the equation (2.29.) is Hill's equation and its solutions (2.19) are periodic. This leads to the following:

Theorem 2.5. The Hill's equation (2.29.) has solutions which are products of two periodic and two oscillating functions. These solutions are:

$$\begin{cases} y_1=\Pi_1(x)\cos\alpha x\\ y_2=\Pi_2(x)\sin\alpha x.\end{cases} \qquad (2.38.)$$

Therefore, there are two amplitudes $|\Pi_1(x)|$ and $|\Pi_2(x)|$ and two periods 2π and $\frac{2\pi}{\alpha}$.

The theorem 2.5. is known as Floquet's theorem. Floquet has written the solutions (2.38.) in the classical form:

$$\begin{cases} y_1=e^{i\alpha x}\cdot\Pi_1(x),\\ y_2=e^{i\alpha x}\cdot\Pi_2(x),\end{cases} \qquad (2.39.)$$

where α is an independent constant, while $\Pi_1(x)$ and $\Pi_2(x)$ are periodic functions.

III. Conclusion

In the oscillations of the second and higher order, both linear and non-linear, it is important to construe the methodology for determination of amplitudes. However, since oscillations are frequently the whole functions, their growth level, i.e. the amplitudes directly depend on Sturm's zeros. It means that issues of amplitudes and issues of zeros are basic interrelated issues, both being of high importance. Differential equations are applied in different scientific areas and an example of this aplication is shown in papers [2-5, 12-13]. In this work, we have determined one-amplitudinal and two-amplitudinal solutions for the canonical equation (1.1.), when it is oscillating and when it can be solved by applying the method of iteration sequences. This concept can be very important for special differential equations, the solutions of which are special functions, such as:

Bessel's equation: $x^2 y'' + xy' + (x^2 - \upsilon^2) y = 0$, $\mathrm{Re}\,\upsilon \geq 0$,

- Hypergeometric (Gaussian) equation: $x(x-1)y'' + [-\gamma + (\alpha + \beta + 1)x]y' + \alpha\beta \cdot y = 0$, $\alpha, \beta, \gamma \in R$ and γ is neither zero nor whole negative number,
- Legendre's equation: $(x^2 - 1)y'' + 2xy' - n(n-1)y = 0$,
- Chebyshev's equation: $(x^2 - 1)y' + xy' - n^2 y = 0$, $n \in R$,

The equation whose coefficients are polynomials with real zeros: $P_{n-2}(x)y'' + R_{n-1}(x)y' + Q_n(x)y = 0$

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