ANNALS OF THE ORADEA UNIVERSITY
Fascicle of Management and Technological Engineering
ISSUE #2, SEPTEMBER 2013, http://www.imtuoradea.ro/auo.fmte/

ON MCSHANE EQUIINTEGRABLE SEQUENCES IN LOCALLY CONVEX SPACES

Ismet TEMAJ, Agron TATO
University of Priština, Priština, e-mail: itemaj63@yahoo.com

***Abstract*—**In this paper, we define the equiintegrability sequence for locally convex space- valued McShane integrable functions and prove the theorem limit for the McShane integrale of functions whose range is a Hausdorff locally convex topological vector space.

***Keywords*—**Equiintegrable sequences, Mcshane integrale, convex space.

I. Introduction and Preliminaries

TRADITIONAL Riemann integration, while powerful, leaves us with much to be desired. The class of functions that can be evaluated using Riemann's technique, for example, is very small. Another problem is that a convergent sequence of Riemann integrable functions (we will denote this class of functions as $R-integrable$) does not necessarily converge to an $R-integrable$ function .Gordon [2] generalized the definition of the McShane integrale for real-valued functions to functions taking values in Banach spaces and investigated some of its properties. Many authors have studied McShane integral (cf.[2], [6]). In this paper, we present convergence theorems for the M-integral of functions taking values in a locally convex space. These theorems are based on the concept of equiintegrability sequence.

The typical form of a limit theorem for an $\aleph$-integral through measure μ is:

- If sequence (f_n) is the $\aleph$- integrable functions sequence, for which the following statements are true:

(a) $\lim_{n\to\infty} f_n(t) = f(t)$ Almost everywhere regarding μ,

(b) statement Q,

then the function f is $\aleph$- integrable, moreover we have that:

$$\lim_{n\to\infty}(\aleph)\int_I f_n(t)d\mu = (\aleph)\int_I f(t)d\mu$$

The Limit theorems for particular type of integrals are important because we can use them in more powerful mathematical techniques for integral studies.

The Lebesgue integral (Dunford) is a type of integral with most powerful theorems of convergence (Limit); statement Q is in fact about limitation of the sequence (f_n) with a function g, which is Lebesgue – integrable and exactly the statement Q can be replaced with: $(\forall n\in N)[|f_n(t)|\le g(t)$ almost everywhere, regarding μ].

Many definitions on Banach space are well-known, by using [7], we emphasize here some relatively new definitions on locally convex spaces, which are simple generalization of the Banach space case. We follow definitions from [3],[4] and [1].

We consider functions $f: I\to X$ where $I\subset \mathrm{R}$ is a compact interval, and let X be a Hausdorff locally convex space (briefly a locally convex space) with its topology τ $((X,\tau))$. $P(X)$ denotes a family of τ-continuous seminorms on X so that the topology is generated by $P(X)$; for every $p\in P(X)$, $\tilde{X}^p$ denotes a quotient vector space of the vector space X with respect to the equivalence relation $x\sim^p y \Leftrightarrow p(x-y)=0$; the map $\phi_p : X\to \tilde{X}^p$ is a canonical quotient map, thus $\phi_p(x)$ is an equivalence class of an element $x\in X$ with respect to the relation "$x\sim^p y$"; a quotient normed space $(\tilde{X}^p,\tilde{p})$ is called a normed component of the space (X,τ), where $\tilde{p}(\phi_p(x)) = p(x)$, for each $x\in X$; a Banach space $(\bar{X}^p,\bar{p})$, which is the completion of the space $(\tilde{X}^p,\tilde{p})$, is called a Banach component of the locally convex space (X,τ); X', X'_p, $\tilde{X}'_p$ and $\bar{X}'_p$ are topological duals of (X,τ), (X,p), $(\tilde{X}^p,\tilde{p})$ and $(\bar{X}^p,\bar{p})$, respectively; $\sigma(X,X')$ is the topology of (X,τ). It is easy to see that

$X' = \{\tilde{x}'_p \circ \phi_p \,/\, \tilde{x}' \in \tilde{X}'_p, p \in P(X)\}$ because for every $x' \in X'$, we have that $|x'(.)| \in P$

The function:

$$\phi : X \to \prod_p (\tilde{X}^p, \tilde{p}), \quad \phi(x) = (\phi_p(x)), x \in X$$

is clearly linear , and since (X, τ) is Hausdorff, it is also one to one. Moreover, the function ϕ is readily seen to be homeomorphism , and hence, an isomorphism of X onto $\phi(X)$ (for the isomorphicdefinition of topological vector space see, [8,p11].

By μ let the Lebesgue measure in R be denoted.

A system (finite collection) of point-interval pairs $\{(I_i, t_i) : i = 1, 2, ..., r\}$ is called an $M-system$ in I if I_i are non-overlapping

$\text{int} I_i \cap intI_j = \phi$ for $i \neq j$, $\text{int} I_i$ is the interior of I_i) , t_i are arbitrary points in I .

An $M-system$ in I is called an $M-$ *partition of* I if $\bigcup_{i=1}^{r} I_i = I$

Given $f : I \to X$ and partition $P = \{(I_i, t_i) : i = 1, ..., r\}$ in I , we set

$$S(f, P) = \sum_{i=1}^{r} f(t_i)\mu(I_i)$$

and call this number the Riemann sum, of f associated with P .

Given $\delta : I \to (0, +\infty)$, called a gauge, an $M-system$ $\{(I_i, t_i) : i = 1, 2, ..., r\}$ in I is called $\delta-$ *fine* if

$$I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i)) \ , i = 1, 2, ..., r .$$

Definition 1. A function $f : I \to X$ is said to be *McShane intergable* (briefly M- integrable) on I , if there exists a vector $\omega \in X$ satisfying the following property: given $\varepsilon > 0$ and $p \in P(X)$ there exists a gauge $\delta_p : I \to (0, +\infty)$ such that for every $\delta_p -$ *fine* $M-$ *partition* $P = \{(I_i, t_i) : i = 1, ..., r\}$ of I the inequality

$$p(S(f, P) - \omega_f) < \varepsilon$$

holds. Denote: $\omega = (M)\int_I f(t) d\mu_L$ and M denotes the set of all McShane integrable functions.

Given a set $E \subset I$ we denote by χ_E its characteristic function

$$(\chi_E(t) = 1 \text{ for } t \in E \ , \chi_E(t) = 0 \text{ otherwise}).$$

A function $f : I \to X$ is called *McShane integral over over the set* $E \subset I$ if the function $f \cdot \chi_E : I \to X$ is *McShane intergable.*

In the case we write $\int_I f \cdot \chi_E = \int_E f$.

II. A LIMIT THEOREM FOR MCSHANE INTEGRAL

Let us consider the problem for a limit theorem regarding M-integral. Is it possible to assume that for this type of integral only the verity of statement (a) is sufficient? The answer is negative and the following example will prove this.

Example1. For every $k \in N$ we define the function:

$$f_k : [0,1] \to \text{R}, \ f_k(x) = k.x_{(0,\frac{1}{k})}(x), x \in [0,1].$$

We can see that the function sequence (f_k) converges in $[0,1]$ to the function $f = 0$ and at the other hand we have:

$$(M) \int_{[0,1]} f_k(s) d\mu_L = 1$$

for $k \in N$ and $(M) \int_{[0,1]} f(s) d\mu_L = 0.$ So, we have:

$$\lim_{k \to \infty} (M) \int_{[0,1]} f_k(s) d\mu_L \neq (M) \int_{[0,1]} f(s) d\mu_L.$$

One limit theorem for M-integral in R is given in [9], in which, instead of the statement *Q* is another statement that has to do with the uniformity of the M – integration of the function sequence (f_n) or equiintegrability of a function sequence (f_n). In [5] is proved the limit theorem for M – integral in Banach space. The essential idea is that by using the topological relations between a locally convex space and its components to extend the well-known results for integrals from Banach space to a locally convex space.

From now on X will be a complete locally convex space.

We need the following lemma

In a locally convex space, according to equality:

$$\tilde{p}(S(\phi_p \circ f, P) - \phi_p(\omega_f)) = p(S(f, P) - \omega_f)$$

and definition 1. it is proved the following lemma

Lemma 1. The function $f : I \to X$ is the M-integrable in a locally convex space if and only if there exists $\omega_f \in X$ such that for every $p \in P(X)$ the function $\phi_p \circ f$ is the M-integrable in the normed component $(\tilde{X}^p, \tilde{p})$ and

$$(M)\int_I \phi_p \circ f(t) d\mu_L = \phi_p(\omega_f)$$

Let us start with the following .

Definition 2. A collection B of functions $f: I \to X$ is called equiintegrable if every $f \in B$ is McShane integrable and for every $\varepsilon > 0$ and $p \in P(X)$ there is a gauge δ such that for any $f \in B$ the inequality

$$p(S(f,P)-(M)\int_I f(t)d\mu_L) < \varepsilon$$

holds provided $P = \{(I_i, t_i): i = 1,...,r\}$ is $\delta_p - fine$ $M - partition$ of I.

Now let us prove the limit theorem for McShane integrale .

The following statement is a clear analog statement of the convergence theorem for HK- integral in the Banach space (se [7] ; T III 5.2)

Main result

Theorem 1. Assume that $B = \{f_n : I \to X \,/\, n \in N\}$ is an equiintegrable sequence such that

$$\lim_{n\to\infty} f_n(t) = f(t), \quad t \in I. \qquad (1)$$

Then the function $f: I \to X$ is McShane integrable and

$$\lim_{n\to\infty}(M)\int_I f_n(t)d\mu_L = (M)\int_I f(t)d\mu_L \qquad (2)$$

holds.

Proof. Let $p \in P$ and $\varepsilon > 0$ be given.. We are going to prove the sequence $((M)\int_I f_n(s)ds)$ is Couchy sequence in (X,τ) .Then according to definition 2., there can be found a gauge $\delta_p : I \to (0,+\infty)$ such that for every $\delta_p - fine$ $M - partition$ $P_0 = \{(I_i, t_i): i = 1,...,k\}$ of I the inequality :

$$p(S(f_n,P)-(M)\int_I f_n(s)ds) < \frac{\varepsilon}{3} \qquad (3)$$

is true for any $n \in N$. If the partition $P_0 = \{(I_i, t_i): i = 1,...,k\}$ is fixed then the pointwise convergence (1) yields

$$\lim_{n\to\infty} S(f_n, P_0) = S(f, P_0) \qquad (4)$$

So we have:

$$p(S(f_n,P_0)-S(f_m,P_0)) = p(\sum_{i=1}^{k} f_n(t_i)\mu(I_i) - \sum_{i=1}^{k} f_m(t_i)\mu(I_i)) \le \sum_{i=1}^{k} p(f_n(t_i)-f_m(t_i))\mu(I_i)$$

For each t_i , there exists a positive integer $K_i(t_i)$ such that

$$p(S(f_n,t_i)-S(f_m,t_i)) \le \frac{\varepsilon}{3k} \quad \text{for all } m,n \ge K_i.$$

Set $N = \max\{K_i \,/\, 1 \le i \le k\}$. Then

$$P(S(f_n,P_0)-S(f_m,P_0)) < \frac{\varepsilon}{3} \qquad (5)$$

for all $m,n \ge N$.

Then according to (3) and (5) we have that:

$$p((M)\int_I f_n(s)ds-(M)\int_I f_m(s)ds) \le p((M)\int_I f_n(s)ds - S(f_n,P)) + p(S(f_n,P)-S(f_m,P)) +$$
$$p(S(f_n,P)-(M)\int_I f_m(s)ds) < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$$

for all $m,n \ge N$. It follows that $((M)\int_s f_n(t)d\mu)$ is a Cauchy sequence in X and as a result there exists $\omega_f \in X$ such that the equality:

$$\lim_{n\to\infty}(M)\int_I f_n(t)d\mu_L = \omega_f$$

holds in this space.

Hence , according to lemma 1., the equality:

$$\lim_{n\to\infty}(M)\int_I \phi_p \circ f_n(t)d\mu_L = \lim_{n\to\infty}\phi_p((M)\int_I f_n(t)d\mu_L) = \phi_p(\omega_f)$$
(6)

holds in normed component $(\tilde{X},\tilde{p})$ and consequently in the Banach component $(\overline{X},\overline{p})$. Also, the sequence $(\phi_p \circ f_n)$ is M- equiintegrable in and consequently in the Banach component and it converges to in and consequently in. As a result, according to theorem 1. (see [5]), the function is M- integrable in and the equality:

$$\lim_{n\to\infty}(M)\int_I \phi_p \circ f_n(t)d\mu_L = (M)\int_I \phi_p \circ f(t)d\mu_L$$

holds and from here, according to (6) , the equality:

$$(M)\int_I \phi_p \circ f(t)d\mu_L = \phi_p(\omega_f)$$

olds. Since $\omega_f \in X$, then the function $\phi_p \circ f$ is M-integrable in $(\tilde{X},\tilde{p})$. Hence , according to Lemma.1, the function f is M- integrable in (X,τ) and

$$\lim_{n\to\infty}(M)\int_I f_n(t)d\mu_L = \omega_f = (M)\int_I f(t)d\mu_L$$

holds.

To proved a theorem convergence for functions taking values in a locally convex space such that its weak topology is complete by sequence, we need the following lemmata.

Lemma 2. Let (X,τ) be a locally convex spaces such that the weak topology $\sigma(X,X')$ is complete by sequence. If the sequence $(f_n : I \to X)$ is M-equiintegrable in X and converges to the function

$f: I \to X$ by weak topology $\sigma(X, X')$ then there exists $\omega_f \in X$ such that the equality:

$$\lim_{n\to\infty} x'((M)\int_I f_n(t)d\mu_L) = x'(\omega_f)$$

holds for every $x' \in X'$.

Proof. Let us denote by P' the family of all continuous semi norms in the locally convex space $(X, \sigma(X, X'))$. Since $P' \subset P$, then the sequence (f_n) is M- equiintegrable in $(X, \sigma(X, X'))$ and converges to the function $f: I \to X$ in this space. So that, we are in conditions of theorem 1. Hence there exists $\omega_f \in X$ such that:

$$\lim_{n\to\infty} p'((M)\int_I f_n(t)d\mu_L - \omega_f) = 0$$

for every $p' \in P'$ and as a consequence we have:

$$\lim_{n\to\infty} x'((M)\int_I f_n(t)d\mu_L) = x'(\omega_f)$$

for every $x' \in X'$, because $|x'(.)| \in P'$.

Lemma 3. If a function $f: I \to X$ is M- integrable in locally convex space X, then for every $x' \in X'$ the function $x' \circ f$ is M- integrable in $(R, |.|)$ and

$$(M)\int_I x' \circ f(t)d\mu_L = x'((M)\int_I f(t)d\mu_L) \qquad (7)$$

Proof. There are given $x' \in X'$ and $\varepsilon > 0$. By the Functional Analysis, there exist a real number $r > 0$ and continuous semi norm p_0 such that inequality:

$$|x'(x)| < r \cdot p_0(x) \qquad (8)$$

holds for $x \in X$ every.

Since the function f is M- integrable in X, then for ε/r and p_0 there exists a guauge $\delta_{p_0}(\varepsilon/3)$ in I such that inequality:

$$p_0(S(f,P) - (M)\int_I f(t)dt) < \frac{\varepsilon}{3}$$

holds for every $\delta_{p_0}(\varepsilon/3) - fine$ $M - partition$ $P = \{(I_i, t_i) : i = 1,...,k\}$ of I and hence , by using the inequality (8) , the inequality:

$$|S(x' \circ f, P) - x'((M)\int_I f(t)dt)| = |x'(S(f,P) - (M)\int_I f(t)dt)| < \varepsilon$$

holds for every $\delta_{p_0}(\varepsilon/3) - fine$ $M - partition$ $P = \{(I_i, t_i) : i = 1,...,k\}$ of I. This means that $x' \circ f$ is M- integrable in $(R, |.|)$ and the equality (7) is true.

Theorem 2. Assume that $B = \{f_n : I \to X / n \in N\}$ is an equiintegrable sequence in X and converges to a function $f: I \to X$ in the weak topology $\sigma(X, X')$, then the function f is M-integrable on I in X and we have

$$\lim_{n\to\infty}(M)\int_I f_n(t)d\mu_L = (M)\int_I f(t)d\mu_L \qquad (9)$$

in the weak topology $\sigma(X, X')$.

Proof. Let p be any continuous semi norm in X. It is denoted X'_p, $\tilde{X}'_p$ and $\bar{X}'_p$ the topologically dual of the space (X, p), $(\tilde{X}^p, \tilde{p})$ and $(\bar{X}^p, \bar{p})$ respectively. Since the sequence (f_n) converges to f in X by weak topology, then the sequence $(\phi_p \circ f_n)$ converges to $\phi_p \circ f$ in the normed component $(\tilde{X}^p, \tilde{p})$ by weak topology, because

$$\tilde{x}^p \circ \phi_p = x'_p \in X'_p \subset X'$$

for every $\tilde{x}'_p \in \tilde{X}'_p$, and consequently the sequence $(\phi_p \circ f_n)$ also converges to the function $\phi_p \circ f$ in the Banach component $(\bar{X}^p, \bar{p})$ by weak topology. So that , we have the sequence $(\phi_p \circ f_n)$ is M- equiintegrable in $(\tilde{X}^p, \tilde{p})$ and converges to $\phi_p \circ f$ in $(\bar{X}^p, \bar{p})$ by weak topology. Then according to theorem 1.([4]) , the function $\phi_p \circ f$ is M- equiintegrable in $(\bar{X}^p, \bar{p})$ and equality:

$$\lim_{n\to\infty} \bar{x}'_p((M)\int_I \phi_p \circ f_n(t)d\mu_L) = \bar{x}'_p((M)\int_I \phi_p \circ f(t)d\mu_L)$$

holds for every $\bar{x}'_p \in \bar{X}'_p$, and since every $\bar{x}'_p \in \bar{X}'_p$ is the continuous extension of an element $\tilde{x}'_p \in \tilde{X}'_p$, then it follows that :

$$\lim_{n\to\infty} \tilde{x}'_p((M)\int_I \phi_p \circ f_n(t)d\mu_L) = \tilde{x}'_p((M)\int_I \varphi_p \circ f(t)d\mu_L) \qquad (10)$$

for every $\tilde{x}'_p \in \tilde{X}'_p$, where $\bar{x}'_p$ is the continuous of $\tilde{x}'_p$. By applying lemma 3. , for every $\phi_p \circ f_n$, we obtain:

$$\tilde{x}'_p((M)\int_I \phi_p \circ f_n(t)d\mu_L) = (M)\int_I (\tilde{x}'_p \circ (\phi_p \circ f_n))(t)d\mu_L) = (M)\int_I (x'_p \circ f_n(t)d\mu_L)$$

where $x'_p = \tilde{x}'_p \circ \phi_p$, and again by applying lemma 3., for every f_n, we obtain:

$$(M)\int_I x'_p \circ f_n(t)d\mu_L = x'_p((M)\int_I f_n(t)d\mu_L)$$

and as a result:

$$\tilde{x}'_p((M)\int_I \phi_p \circ f_n(t)d\mu_L) = x'_p((M)\int_I f_n(t)d\mu_L) \quad (11)$$

Hence , by replacing the right-hand-side of (11) to (10), we have:

$$\lim_{n\to\infty} x'_p((M)\int_I f_n(t)d\mu_L) = \overline{x}'_p((M)\int_I \phi_p \circ f(t)d\mu_L) \quad (12)$$

Also, according to lemma 2. , there exists $\omega_f \in X$ such that the equality:

$$\lim_{n\to\infty} x'_p((M)\int_I f_n(t)d\mu_L) = x'_p(\omega_f) = \tilde{x}'(\phi_p(\omega_f)) \quad (13)$$

holds for every $x'_p \in X'_p$, where $\overline{x}'_p$ is the continuous of $\tilde{x}'_p$. Consequently:

$$\overline{x}'_p(\phi_p(\omega_f)) = \overline{x}'_p((M(\int_I \phi_p \circ f(t)dt)$$

for every $\overline{x}'_p \in \overline{X}'_p$, which means that:

$$(M)\int_I \phi_p \circ f(t)d\mu_L = \phi_p(\omega_f) \in \tilde{X}^p$$

and hence , by lemma1. , the function f is M-integrable in X and

$$(M)\int_I f(t)d\mu_L = \omega_f$$

REFERENCES

[1] Blondia, C., Integration in locally convex spaces, Simon Steven 55 (1981) 81-102.

[2] Gordon R. A,The McShane integral of banach- valued functions, Illinois J. Math. 34 (1990), no. 3, 557-567.

[3] Marraffa, V., Nonabsolut convergent integrals of functions taking values in a locally convex space. Rocky Mountain Journal of Mathematics, Volume 36, Number 5 (2006), 1577-1593.

[4] Memetaj, S., Integration topics on locally convex spaces (in Albanian), Mokra, Tirana, 2007

[5] Temaj, I., Tato, A., On McShane equiterable sequences, Journal of Modern Mathematics and Statistics 4 (4): 123-126 2010 , ISSN 1994-5388.

[6] Bartle,G. , A Modern Theory of Integration, Amer. Math. Soc., Providence, 2001.

[7] Shwabik, S., Guoju, Y., Topics in Banach space integration, World Scientific, Singapure, 2005.

[8] Schafer, H.H. , Topologicl vector Spaces. Graduate Texts in mathematcs, Vol3 Springer- Verlog, New Yok 1971.

[9] Schwabik Š., Guoju Y., Topics in Banach space integration World Scientific, Singapure, 2005.