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ON THE INTEGRATION IN MIXED TOPOLOGICAL IN A LOCALLY CONVEX SPACE

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Abstract—In this paper we construct one mixed topology from a natural one with its compatible, based on concepts of inductive and projective topologies. We present some constructive definition for measurability and integrations in mixed topological space. We base it on [H. H. Schaefier, Topological Vector Spaces anad Application to Stochastic integration, Handbook of Measure Theory, Volume I, Elsevier, 2002], and moderate the works of Memetaj [S. Memetaj Tema Integrimi ne Hapesirat lokalisht konvekse, Mokra, Tiran 2007], and practically achieve the same results as in the case of Frechet space.

Keywords—Summable families, mixed topology, locally convex space, Frechet space.

I. Introduction

THIS paper is affected from well-known concepts of summable families developed from Pietsch [5] and their applications to the mixed topologies studied from Cooper[2]. We construct one mixed topology from a natural one with its compatible, based on concepts of inductive and projective topologies. These allow us to give some more descriptive definitions for measurability and integration of the set functions. In the second section we define two topologies and compare them with each-other. In the third section we give some known definitions and extend them for the measurability of functions and we extend some properties of the integration in locally convex Fréchet spaces to the mixed topologies ones. A family of numbers $[\xi_{i.}I]$ is a set of real or complex numbers which are corresponded in an unique way to the elements i of an index directed set I.

We denote the collection of all finite subsets of I with F and with F (I) the family of these subsets. Construct the partial finite sum

$$\sigma_F = \sum_{i\in F} \xi_i , \quad F\in F (I).$$

If the system of these finite sums (σ_F), for every F∈ F (I) is convergent, the family $[\xi_{i.}I]$ is called *summable* and a sum of these partial sums is considered the sum of series. We denote

$$\sigma = \sum_I \xi_i$$

Lemma 1.1 [5, p.18]. *A family of numbers* $[\xi_{i.}I]$ *is summable if and only if there is a positive number* ρ *for which F∈ F (I) the inequality*

$$\sum_F |\xi_i| \leq \rho \qquad (1)$$

is valid.

This condition is valid and for the summability of the family $D[|\xi_i|, I]$.

Let $\{X_i\}$ a family of the seminormed, generated from family of seminorms $\{p_i\}$. It is known that the cartesian product $\Pi = \prod_{i=1}^{r} X_i$ is seminormed from at least two important seminorms $e(x) = \sum_{i=1}^{r} p_i(x)$ and $h(x) = \max_{1\leq i\leq r}\{p_i(x)\}$ and the inequality

$$h(x) \leq e(x) \leq r\cdot h(x) \qquad (2)$$

hold.

We see that two above seminorms are equivalent in a cartesian product.

II. The summable families in locally convex space

Let us recall some definition from locally convex space. The index family $[x_i, I]$ from locally convex space E is called *absolutely summable* if for every neighborhood U of zero, there is the number ρ such that stets the relation

$$\bigcup_I p_U(x_i) < \rho$$

It is followed immediately from the definition that collection $\{E\}$ of the absolutely summable families $[x_{i.}I]$ is a linear space with the operations:

$[x_i, I]+[y_i, I]=[x_i+y_i, I]$ and $\lambda[x_i, I]=[\lambda x_i, I]$

If we take as seminorm

$$\pi_U[x_{i.}I]=\sum_I p_U(x_i)$$

where U is neighborhood of zero, we generate a family of locally convex spaces the so called π- topollgy *:*

Let us prove by constructing the topologies by means of summable families. In the index collection I is finite, we can create the same topology if take the seminorm

$$h_U[x_{i.}I]=\max_{i\in I}\{p_U(x_i)\}$$

We pretend to extend this property for the infinite case.

Let $\{(\mathrm{E}_\alpha,\ \mathrm{p}_\alpha)\}_{\alpha\in A}$ be the family of the seminormed spaces and denote E the their cartesian product

$$E=\prod_{\alpha\in A} E_\alpha$$

Functions

$$\mathrm{p}_1:\ \mathrm{x}=(x_\alpha)\rightarrow\sum_{\alpha\in A} p_\alpha(x_\alpha)$$

$$p_0: x=(x_\alpha)\rightarrow\sup_{\alpha\in A}\{p_\alpha(x_\alpha)\}$$

are seminorms to the subspaces

$$E_1=\{\mathrm{x}\in\mathrm{E}\ :\ \mathrm{p}_1<\infty\}$$

and

$$E_0=\{\mathrm{x}\in\mathrm{E}\ :\ \mathrm{p}_0<\infty\}$$

Let us prove this only for p_0.

1. $p_0(x)\ge 0,\ p_0(0)=\sup_{\alpha\in A}\{p_\alpha(0)\}$

2.
$$p_0(\lambda x)=\sup_{\alpha\in A}\{p_\alpha(\lambda x)\}=|\lambda|\sup_{\alpha\in A}\{p_\alpha(x)\}=|\lambda|\ p_0(x)$$

3.
$$p_\alpha(x+y)\le p_\alpha(x)+p_\alpha(y)\le\sup_{\alpha\in A}\{p_\alpha(x)+\sup_{\alpha\in A}\{p_\alpha(y)$$

since

$$p_0(x+y)=\sup_{\alpha\in A}\{p_\alpha(\lambda x)\}\le\sup_{\alpha\in A}\{p_\alpha(x)\}+\sup_{\alpha\in A}\{p_\alpha(y)\}$$

We show that from the summability of the family $[p_\alpha, I]$ comes that the p_0 is boarded. We use the finite case to arrive to the converse preposition. We need Zorn lemma for this.

Lemma 2.1 *If the family* $S=[p_\alpha, A]$ *is summable. There exists the real number* θ *so that*

$$\sup_{\alpha\in A}\{p_\alpha(x)\}\le\theta. \qquad (3)$$

We consider only such α *that* $\alpha\in F$ *and cartesian product* $\prod_\alpha E_\alpha$ *.By the inequations (2) we have*

$$\max_{\alpha\in F}\{p_\alpha(x)\}\le\sum_F p_\alpha\le\rho$$

Let us see the family of finite subsets $F\subset A$ for fixed x from the set E. The set of $\{p_\alpha\}$ as a set of finite ordered real numbers from $[p_\alpha, A]$ has the maximum $\max_{\alpha\in A}\{p_\alpha(x)\}$. From the Zorn lemma there exists the number θ so that

$$\sup_{\alpha\in A}\{p_\alpha(x)\}\le\theta$$

Let (p_n) be a sequence of seminorms from the family Σ that for them holds $p_n\le p_0$. Let (λ_n) be a sequence of positive real numbers converges to the $+\infty$, it is easy to prove that $\sum_{i=1}^r\frac{p_i}{\lambda_i}$ is seminorm in the cartesian product respective.

Preposition 2.2 *If for every* $\alpha\in A$ *inequality (3) holds, then the family* $[p_i, N]$ *is summable.*

Proof. Let

$$\{F_k\}_{k\in N} \qquad (4)$$

be an increasing sequence of sets where the index are ordered in natural numbers, so $F_k=\{i_1, i_2,\cdots i_k\}$ and i_t for $s<t$. Denote with $F_{n,k}$ the finite set of indexes $F_{n,k}=F_k\setminus F_n$. Consider the sum

$$\frac{1}{\lambda_{n,k}}\sum_{i\in F_{n,k}} p_i$$

where $\lambda_{n,k}=card(F_{n,k})$. We can see that $(\lambda_{n,k})$ is the sequence that converges to $+\infty$ where $n-k\rightarrow+\infty$. Apply the equality (2) for the respective cartesian product we take

$$\sum_{i\in F_{n,k}} p_i\le card(F_{n,k})\max_{i\in F_{n,k}}\{p_i\}$$

or

$$\frac{1}{card(F_{n,k})}\sum_{i\in F_{n,k}} p_i\le\max_{i\in F_{n,k}}\{p_i\}\le\theta$$

This proves that the family $[\frac{p_i}{\lambda_k}, N]$ is summable by suitable choosing of λ_k . In order to prove that the family $[p_i, N]$ is summable we suppose the contrary. By this option, there exists the positive number r and the increasing sequence of sets

$$F_1, F_2,\cdots, F_n,\cdots$$

and seminorms p_i for which holds

$$\sum_{i\in F_{n,k}} p_i \geq r^2$$

If we check the summable family $[\frac{p_i}{\lambda_k}, N]$ and take the finite sum, where again λ_k are chosen as $\lambda_{n,k} = card(F_{n,k})$ and $\{F_{n,k}\}$ are from sequence (4), we take

$$\sum_N \frac{1}{\lambda_i} p_i \geq \sum_{i\in F_r} \frac{1}{\lambda_k} p_i \geq \sum_{i\in F_r} \frac{1}{r} p_i \geq r.$$

We take this important preposition for the numerable case.

Preposition 2.3 *If the family* $[p_\alpha, A]$ *is summable then the seminorms* p_0 *and* p_1 *are equivalent in the cartesian product of the seminormed spaces.*
Denoted with E_1 *and* E_0 *respectively* $M\sum_{\alpha\in A} E_\alpha$ *and* $M\prod_{\alpha\in A} E_\alpha$. *By virtue of above seminorms this space are seminormed space , they are complete if every* E_α *is complete.*

We consider the two spectra functions so called contract $\{\pi_{\alpha\beta} : E_\beta \to E_\alpha, \alpha \leq \beta,\ \alpha,\beta \in A\}$, where $\pi_{\alpha\alpha} = id(E)$ for every α and if $\alpha \leq \beta \leq \gamma$ then $\pi_{\gamma\alpha} = \pi_{\beta\alpha} \circ \pi_{\gamma\beta}$.

As the second spectra we take the function $\{i_{\alpha\beta} : E_\alpha \to E_\beta, \alpha \leq \beta,\ \alpha,\beta \in A\}$ which is the linear contract with $i_{\alpha\alpha} = id(E_\alpha)$ and for $\alpha \leq \beta \leq \gamma$ we have $i_\gamma = i_{\beta\gamma} \circ i_{\alpha\beta}$.

Following the arguments given from [2] and [6] in case of the normed space we have:

Proposition 2.4 *The spaces* E_1 *and* E_0 *are respectively subspaces of projective and inductive limes of the space* E_α *considering only linear contracts.*

Proof. a) Let A ordered set. The projective limes of spectra π is the closed subspace of $M\sum_{\alpha\in A} E_\alpha$, we denote it with $M - \lim\limits_{\leftarrow}\{E_\alpha, \pi_{\beta\alpha}\}$, and is contained from $\{(x_\alpha)\} \in M\prod_{\alpha\in A} E_\alpha$, where $\pi_{\alpha\beta}(x_\beta) = x_\alpha$ for $\alpha \leq \beta\}$

b) Inductive limes of this spectra is space taken as the factor space of $M\sum_{\alpha\in A} E_\alpha$ in relation with a subspace generated from the elements of the form $(x_\gamma - i_{\beta\gamma}(x_p))$, if we consider every space E_β as a subspace of $M\sum_{\alpha\in A} E_\alpha$, taken in suitable manner .

We can prove the preposition also from Samedini [7] arguments.

We give the inductive limes another face. Suppose the E_α is a closed subset of one seminormed space F and A is a ordered set such way that $\alpha \leq \beta$ if and only if $E_\alpha \subseteq E_\beta$. This involves that $i_{\alpha\beta}$ to be a natural injective function. In this case, inductive limes is isomeric with the closure of $M\sum_{\alpha\in A} E_\alpha$ in F .

Let E be a set where its topology is endowed with the a family seminorms [Γ,A] which are uniformly bounded. This mean that for every $\alpha \in A$ and $x \in E$, $p_\alpha(x) \leq K$ holds. Denote τ the topology generate from this family of seminorms. We consider that family separate the points (total family). It is well-known that if we choose as neighborhood of zero the family $V_{\varepsilon,F} = \{x \in E :$ F finite set from A and for every $i \in F, p_i(x) < \varepsilon\}$ we have locally convex topology.

Denote $p_0(x) = \sup_{\alpha\in A}\{p_\alpha(x)\}$. It easy to show that this function is seminorm in $E_0 = E = \{x : p_0(x) \leq m\}$. Let x, y be two arbitrary points of E .Since the function p_α are convex for every α

$$p_\alpha(\lambda x + (1-\lambda)y \leq \lambda p_\alpha(x) + (1-\lambda)p_\alpha(y) \leq m \ .$$

This seminorm endowed on E one other topology different from τ , we denote this new topology with η . So we have two topologies in the same set E . In this case for the space E we have the so called mixed topology and denote (E, p_0, τ) .

Preposition 2.5 *The topologies* τ *and* η *are compatible .*

Proof. We must prove that two topologies are comparable. Let V be a neighborhood of zero in the space (E,η). We can write $V = \{x \in E : p_0(x) < \varepsilon\}$. Since for every $\alpha \in A$ we have $p_\alpha(x) < \varepsilon$, then $x \in V \subset V_{\varepsilon,\alpha}$ or $V \subset V_{\varepsilon,F} = \bigcap_{\alpha\in F} V_{\varepsilon,\alpha}$. It proves that every open set of τ is open set of η . The same is also inverse.

Although the triad (E, p_0, τ) doesn't contain a normed space compatible in explicit form, we know that the topology η generated from p_0 is more richer than locally convex topology τ, so we can consider the mixed topology (E, p_0, τ) as a topology of Sax type .

Preposition 2.6 *The following properties hold*

1) *Every closed set in* (E,τ) *is closed in* (E,η)
2) *For a set* $A \subset E$, η*-closure and* τ*-closure have the relation* $cl_\eta(A) \subseteq cl_\tau(A)$.
3) *A bounded set in* (E,η) *(totally bounded) is bounded (totally bounded) in* (E,τ).
4) *A Cauchy net in* (E,η) *is a Cauchy net in* (E,τ).
5) *A net is convergent to* x_0 *by* τ *if it is converge to* x_0 *by* η.
6) *If the space is complete (sequential complete)* (E,η) *so is* (E,τ).

Proof. 1) This is evident from preposition 2.5
2) Let $x \in cl_\eta(A)$ signify that for every neighborhood of zero in this space $(x+V_0)\cap A \neq \phi$. Since every neighborhood of zero $V_{\varepsilon,F}$ of the space (E,τ) contains this neighborhood for every $\alpha \in A$, we have that $(x+V_{\varepsilon,F})\cap A \neq \phi$ on (E,τ). This shows that $cl_\eta(A) \subset \bigcap_{\alpha\in A} cl_\alpha(A)$, where $cl_\alpha(A)$ is denoted the closure of A in respect of topologies (E, p_α). We take our proof in regard to [8] and [4] where it is proved that

$$cl_\tau(A) = \bigcap_{\alpha\in A} cl_\alpha(A)$$

3) By virtue of [1], regarding that the function $h: E \to L = \prod_{p_\alpha\in A}(E, p_\alpha)$. $h(x) = (x)_\alpha$ is homeomorphism of E into $h(E)$. It follows that function $pr_\alpha \circ h$ of (E,τ) into (E, p_α) is continuous. It implies that a set is bounded (totally bounded) on (E,τ) if and only if this is bounded (totally bounded) on every (E, p_α). Since the function p_0 is continuous and convex , we have that for the bounded (totally bounded) set K the set $p_0(K)$ is bounded in R, it follows that all the sets $p_\alpha(K)$ are bounded so is and K on (E,τ). The other properties can be proved in the same manner as the above ones. We note that in [4] is proved that these properties hold in (E,τ) if and only if they hold in every (E, p_α). The inequation

$$p_\alpha(x_n - x_m) \leq p_0(x_n - x_m) < \varepsilon$$

realize that these sequences (nets) are sequences (nets) Cauchy in every space (E, p_α) and this brings the proof.

Following the terminology from Cooper [2] we have the definition.

Definition 2.7 ([2]). Saks space is called the *triplet* $(E, \|\cdot\|, \tau)$: when E is vectorial space, τ is a locally convex topology in E and $\|\cdot\|$ is a norm in E and satisfy the equivalent conditions

a) Unit ball $B_{\|\cdot\|}$ is closed in (E,τ)

b) $\|\cdot\|$ is lower semi-continuous for τ

c) $\|\cdot\| = \sup\{p : p$ is a τ-continuous seminorm with $p \leq \|\cdot\|\}$

Is easy to see that our mixed topology fulfills the third condition not for a norm but for a seminorm, so we construct a new type of Saks space by extending the first one.

III. Integration in Mixed Topology

We present some constructive definition for measurability and integrations in a mixed topological space. We base it on [3, 6], and moderate the works of Memetaj [4], and practically achieve the same results as in the case of Frechet space.

Let us suppose that (S, Σ, μ) is a measure space where μ is a nonnegative complete and finite measure and (E,τ) locally convex space generate from above continuous seminorms. Let us recall some well-known definition for measurability and integral -ability in these spaces.

1) A function $f: S \to E$ is said to be τ*-strongly measurable* if there exists a sequence (f_n) of measurable simple functions so that for every $p_\alpha \in \Gamma$ and every $\varepsilon > 0$ exists n_0 for all $n > n_0$ holds $p_\alpha(f_n(s) - f(s)) < \varepsilon$ μ-a.e. in τ

2) Let $p_\alpha \in \Gamma$. A function $f: S \to E$ is measurable by p_α if there exists the set $S_{0,\alpha} \subset S$ with $\mu(S_{0,\alpha}) = 0$ and the sequence $\left(f_n^\alpha\right)_{n\in N}$ of measurable simple functions such that $\lim_{n\to\infty} p_\alpha(f_n^\alpha(s) - f(s)) = 0$ for every $s \in S \setminus S_{0,\alpha}$. In this case it is said that the function $f: S \to E$ is *measurable* by τ -seminorms if it is measurable by each $p_\alpha \in \Gamma$.

3) A function $f:S\to E$ is said to be η*-measurable* if for every positive real number $\varepsilon>0$ exists n_0 for all $n>n_0$ holds $p_0(f_n(s)-f(s))<\varepsilon$ μ a.e. where $p_0=\sup\{p_\alpha:\alpha\in A\}$.

Immediately we take the preposition.

Preposition 3.1 *If the function $f:S\to E$ is η-measurable then it is τ-srtongly measurable and measurable by seminorms.*

Let us observe the feature of η*-measurability* and compare this with the above measurability. Let us prove in the beginning the visible lemma.

Lemma 3.2 *The subspace of one locally convex separable space is also a locally convex separable space.*

Proof. Let E_0 be a locally convex subspace of locally convex space E, generated from one family of seminorms $\{p\}$ and $E_0\subset E$. Denote A the numerable set of $A=\{a_1,a_2,\cdots\}$ everywhere dense in E. Denote $K_{ij}=K(a_i,\frac{1}{j})=\{x\in E:p(a_i,x)<\frac{1}{j}\}$ the ball with center the point a_i of A and radius $\frac{1}{j}$. Let be $x\in E_0$. If $K_{ij}\cap E_0\neq\phi$ denote with b_{ij} on point of this intersection. Denote with B the set of these points: $B=\{b_n\}$. Show that numerable set and everywhere dense in E_0 . Let x be a point of E_0.Chose an arbitrary $\varepsilon>0$, such that $\frac{2}{j}<\varepsilon$. Since set A is dense on E, there exists a point a_i from A, that $p(a_i,x)<\frac{1}{j}$. It follows that $x\in K_{ij}$ and $p(b_{ij},x)\le p(b_{ij},a_i)+p(a_i,x)<\varepsilon.$

We can prove many properties that come from η-*measurability*. We recall some of them.

Preposition 3.3 *If the $f:S\to E$ is η-measurable function then*

a) The function f is with almost separable value by μ in (E,η) and in every (E,τ_α)

b) For every open set G at (E,τ_α), on has that $f^{-1}(G)$ is open in Σ.

Proof. By the virtue of definition of η*-measurability* of function, there exists the sequence of function f_n and the set $Z_0\in\Sigma$ such that $\mu(Z_0)=0$ and for every $\varepsilon>0$ and $n>n_0$ we have $p_0(f_n(s)-f(s))<\varepsilon$ for $s\in S\setminus Z$. Evidently, the equation hold

$$f(S\setminus Z_0)=\overline{\bigcup_{n\in N}f_n(S\setminus Z_0)}$$

The set $\overline{\bigcup_{n\in N}f_n(S\setminus Z_0)}$ is separable as the closure of a separable set. As the subspace of the separable the set $f(S\setminus Z)$ is separable (lemma 3.2). Consider, preposition 2.6, property 2) we prove that $\overline{A_{p_0}}\subset\bigcap_{\alpha\in A}\overline{A_{p_0}}$. It follows that $f(S\setminus Z)$ is separable in every (E,τ_α).

Let G be an open set in (E,η). Denote G_n the set of elements $x\in G$ so that $R^0_{\frac{1}{n}}(x)\subset G$. It seems clear that $s\notin Z_0$ and $f(s)\in G$ if and only if $f_k(s)\in G_n$ for $k>n_0$. Indeed when $f_k(s)\in G_n$ it implies that $p_0(f_k(s),f(s))<\frac{1}{n}$. That means $f(s)\in G$. On the other side, if $f(s)\in G$, for every $\varepsilon>0$ and $\frac{1}{n}$ exists the natural number n_0 that for $k>n_0$ we have $p_0(f_k(s),f(s))<\frac{1}{n}$. That shows $f_k(s)\in G$. That proof of the first part comes from the equality

$$f^{-1}(G\setminus Z_0)=\bigcup_{m,n=1}^{\infty}\bigcap_{k=m}^{\infty}f_k^{-1}(G_n\setminus Z_0)$$

For the second part, consider implication $Z_0\subset Z_\alpha$ for the every sequence function convergence by seminorms. We see that if G is an open set (E,τ) this is open in (E,η). From the proof of the first part follows $f^{-1}(G)\in\Sigma$ for every component (E,τ_α). The relation between the locally convex spaces and their components are studied carefully from [4], and we use these concepts, and put another function from S to E_0, where E_0 factor space of (E,η). Let (E,τ) be a generated topology from the family of seminorms $\{p_\alpha\}_{\alpha\in A}$. The convex space (E,τ_α) consider as components of the space (E,τ) and topology embedded in this as the inductive topology from these topologies. Denote $Z_\alpha=\{x\in E\ :\ p_\alpha(x)=0\}$. As we know, this is a vectorial subspace . Denote φ_α for every α factor function

$\varphi_\alpha:E\to\tilde{E}_\alpha=E\setminus Z_\alpha$ such that $\varphi_\alpha(x)=\tilde{x}$ for every $x\in E$.

It is know that the continuous function

$\tilde{p}:\tilde{E}_\alpha\to R$, $\tilde{p}(\tilde{x})=p(x)$, for every $\tilde{x}\in\tilde{E}_\alpha$

is a norm on $\tilde{E}_\alpha$ and the normed space $(\tilde{E}_\alpha, \tau(\tilde{p}_\alpha))$ is isometric isomorph with vectorial subspace $\bar{\tilde{E}}_\alpha = \{(\tilde{x})_{n\in N} \in \bar{E} : \tilde{x} \in \tilde{E}\} \subset \bar{E}$. The space $(\tilde{E}_\alpha, \tau(\tilde{p}_\alpha))$ is the dense subspace of Banach space $(\bar{E}_\alpha, \tau(\bar{p}_\alpha))$. Denote with $\varphi_0 : E \to E_0 = E \setminus Z_0$, where $Z_0 = \{x \in E \ : \ p_0(x) = 0\}$ and with $i_\alpha : (E, \tau_\alpha) \to (E, \eta)$. Since $\tau_\alpha \subset \eta$, we have

$$S \xrightarrow{f} E \xrightarrow{\varphi_\alpha} (E_\alpha) \xrightarrow{i_\alpha} E_0$$

The function f is called a *measurable component* (E, τ_α) by the measure μ if there exists the sequence of simple functions (f_n) converged almost uniformly to f, meaning that: for every $\delta > 0$ there exists the set $S \in \Sigma$ such that $\mu(S \setminus S_\delta) < \delta$ and the sequence of function converges uniformly on S_δ.

Preposition 3.4 ([8]) *If the function $f : S \to E$ is measurable by μ on the component (E, τ_α) of the locally convex space (E, τ) and $A \in \Sigma^+$, then for every $\delta > 0$ exists $A_\delta \in \Sigma^+$ such that $\mu(A \setminus A_\delta) < \delta$ and the set $f(A_\delta)$ is totally bounded on (E, τ_α).*

Lemma 3.5 ([4]) *If the function $f : S \to E$ is measurable by μ on component (E, τ_α) of the locally convex space (E, τ), then exists a finite sequence of disjoint sets $(S_k) \subset \Sigma^+$ for which holds*

1) $\mu\left(S \setminus (S_1 \cup S_2 \cup \cdots \cup S_k)\right) < \dfrac{1}{2^k}$

2) The set $f(S_k)$ is totally bounded on (E, τ_α).

Preposition 14. *A function $f : S \to E$ is measurable in (E, η) if and only if it is measurable by μ in every component (E, τ_α).*

Proof. The right part follows from the definition of $\eta - measuarbility$.

Prove the sufficient condition. Denote $T_k = \bigcup_{i=1}^{k} S_k$. Fix $\alpha \in A$ and take the sequence of the sets (T_k^α). Since $f(T_k^\alpha)$ is a totally bounded set , for every $\varepsilon > 0$, there is an open cover with the balls $(R_\varepsilon^\alpha f((s))_{s \in T_k}$ of the set $f(T_k^\alpha)$ from which we can take the finite subcover $(R_\varepsilon^\alpha f((s_i^k))_{i=1}^{r_k}$.

Denote

$$F_i^\alpha = T_k^\alpha \cap (R_\varepsilon^\alpha(f(s_i^k)) \in \Sigma, i = 1, 2, \cdots r_k$$

and

$$E_1^\alpha = F_1^\alpha, E_i^\alpha = F_i^\alpha \setminus (F_1^\alpha \cup F_2^\alpha \cup \cdots \cup F_{i-1}^\alpha).$$

The above sequence is a sequence of disjoint sets and $\bigcup_{i=1}^{r_k} E_i^\alpha = T_k^\alpha$.

From their construction we have

$$(\forall s \in T_k^\alpha)(\exists i_0 \in \{1, 2, \cdots r_k\}(p_\alpha(f(s) - f(s_{i_0}^k)) < \varepsilon)$$

.

Let us construct the functions

$$f_k = \sum_{i=1}^{r_k} f(s_i^k)\chi_{E_i^\alpha} + 0 \cdot \chi_{S \setminus T_k^\alpha}$$

were $(\forall s \in T_k^\alpha)(p_\alpha(f_k(s) - f(s)) < \varepsilon))$.

Prove that the sequence (f_k) converges almost uniformly after μ to the function f on (E, τ_α). This follows from the fact that for every $\delta > 0$ we know $\lim_{k\to\infty} \mu(T_k^\alpha) = 1$.

Prove that sequence of the functions (f_n) converges almost uniformly on space (E, τ). For that denotes $T_0 = \bigcup_{\alpha \in A} T_k^\alpha$ and we have

$$\mu(S \setminus T)_0 = \mu(S \setminus \bigcup_{\alpha \in A} T_k^\alpha) = \mu(\bigcap_{\alpha \in A|} S \setminus T_k^\alpha) \le \delta.$$

To prove that sequence (f_n) converges uniformly to the function $f(x)$ on T_0, we see the inequalities

$$p_0(f_n(s_i^\alpha) - f(s)) - \frac{\varepsilon}{2} < p_\alpha(f_n(s_i^\alpha) - f(s)) < \frac{\varepsilon}{2}$$

or

$$p_0(f_n(s_i^\alpha) - f(s)) < \varepsilon.$$

and this finishes the proof.

Moreover we introduce a descriptive definition of the Bochner integral in the case of locally convex space generated from the family of continuous uniformly bounded seminorms.

Let us recall some important definition of integralability for locally convex spaces.

(I) A function $f : S \to E$ is said to be $\tau - strongly$ (or BV_τ) *integrable* if there exists a sequence $(f_n)_{n\in N}$ measurable simple functions such that :

1. $f_n(s) \to f(s)$ a.e. by μ in topology τ, that is to say f is $\tau - strongly$ measurable.

2. $p_\alpha(f_n(s) - f(s)) \in L^1(\mu)$ for every $n \in N$ and

$$\lim_{n\to\infty}\int_S p_\alpha(f_n(s) - f(s))d\mu = 0 \text{ for every } p_\alpha \in \Gamma.$$

3. $\int_A f_n d\mu$ *converges* on (E,τ) for every $A \in \Sigma$.

In such case we write $\lim_{n\to\infty}\int_A f_n d\mu = (BV_\tau) - \int_A f d\mu$.

(II) A function $f : S \to E$ is said to be *integrable by seminorm* p_α if there exists the subset S_{0,p_α} of S with $\mu(S_{0,p_\alpha}) = 0$ sequence $(f_n)_{n\in N}$ measurable simple functions $(f_n^\alpha)_{n\in N}$ such that

1. For every $s \in S \setminus S_{0,p_\alpha}$ $\lim_{n\to\infty} p\left(f_n^\alpha(s) - f(s)\right) = 0$
2. $\lim_{n\to\infty}\int_S p_\alpha\left(f_n^\alpha(s) - f(s)\right)d\mu = 0$.
3. For every $A \in \Sigma$ exist the element $x_A \in E$ such that

$$\lim_{n\to\infty}\int_S p_\alpha\left(f_n^\alpha(s) - x_A\right)d\mu = 0.$$

Let us introduce another definition for mixed topologies

(III) A function $f : S \to E$ is said to be $\eta - strongly$ (or BV_η) *integrable* if there exists the sequence $(f_n)_{n\in N}$ measurable simple functions $(f_n)_{n\in N}$ such that

1. $f_n(s) \to f(s)$ a.e. by μ in topology η, so function f is η*- measurable*
2. $p_0(f_n(s) - f(s)) \in L^1(\mu)$ for every $n \in N$ and

$$\lim_{n\to\infty}\int_S p_0(f_n(s) - f(s))d\mu = 0.$$

3. $\int_A f_n d\mu$ *converges* at (E,η) for every $A \in \Sigma$.

(the sequence $(f_n)_{n\in N}$ is said to be representative sequence of f.

From the definitions (I)-(III) one has

Preposition 3.7 *If the function* f *is* (BV_η)*-integrable then it is* (BV_τ) *–integrable and integrable by seminorms.*

Preposition 3.8 *If the space* E *is sequential complete and function* f *is* (BV_η) *integrable, sequence* (f_n) *is representative then* $p_0(f)$*) is Bochner integrable.*

Proof. We use the inequality $| p_0(f_n(s)) - p_0(f_m(s)) | \le p(f_n(s) - p(f_m(s))$ and we have

$$\int_S | p_0(f_n) - p_0(f_m) | d\mu \le \int_S p_0(f_n - f_m)d\mu \le \varepsilon \text{ for } m,n > n_0$$

It means that the sequence $(p_0(f_n))$ is the Chauchy sequence. Moreover $\lim_{n\to\infty} p_0(f_n) = p_0(f)$ and we have the representative sequence of $p_0(f)$.

We close our paper with a proposition that holds only for Banach and Frechet space

Preposition 3.9 *A function* f *is* η*-integrable, if and only if, it is integrable by every seminorms* p_α *for every* $\alpha \in A$.

Proof. The necessary condition follows from the definition and preposition 3.8 by proving the sufficient condition. We suppose that the function is inetgrable by every seminorm p_α. Show that the sequence of functions constructed in preposition 3.7 is a representative sequence for the function f on (E,η). We chose a $\beta \in A$ that fulfills the condition $p_0 - \frac{\varepsilon}{2} < p_\beta$ and observe

$$\lim_{k\to\infty}\int_S p_\beta(f_k(s) - f(s))d\mu = \lim_{k\to\infty}\int_{T_0} p_\beta(f_k(s) - f(s))d\mu + \lim_{k\to\infty}\int_{S\setminus T_0} p_\beta(f_k(s) - f(s))d\mu =$$
$$= \lim_{k\to\infty}\int_{T_0} p_\beta(f_k(s) - f(s))d\mu + \lim_{k\to\infty}\int_{S\setminus T_0} p_\beta(f_k(s))d\mu \quad (5)$$

By virtue of inequality $p_\beta(f_k(s) - f(s)) < \frac{\varepsilon}{\mu(T_0)}$ for every $x \in T_0$ we have

$$0 \le \lim_{k\to\infty}\int_{T_0} p_\beta(f_k(s) - f(s)) \le \frac{\varepsilon}{\mu(T_0)} \cdot \mu(T_0) = \varepsilon \quad (6)$$

if we consider that $\lim_{k\to\infty}\mu(S \setminus T_0) = \lim_{k\to\infty}\mu(S \setminus \bigcup_\alpha T_k^\alpha) = 0$. Since the for every α seminorm p_α is Bochner (Lebeg) integrable and is continuous we have

$$\lim_{k\to\infty}\int_{S\setminus T_0} p_\beta(f_k(s) - f(s))d\mu = 0 \quad (7)$$

If we substitute (6) and (7) to (5), and consider that sequence (f_k) uniformly converges after μ to f we have that sequence (f_k) is a representative sequence of f on (E,η) .

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