EXPERIMENTAL FINDING OF AXIAL MOMENT OF INERTIA

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***Abstract*—**The paper presents a method proposed for estimating the axial moment of inertia of a rigid body. The principle of the method consists in finding the oscillation periods for physical pendulum created by suspending the considered body by bearings with axes parallel to the axis considered for finding the axial moment of inertia. These data are sufficient to estimate both the central moment of inertia with respect to an axis parallel to the considered axis and the position of the center of mass.

***Keywords*—**vector dynamics, axial moment of inertia, physical pendulum period

I. INTRODUCTION

THE dynamic behaviour of a mechanical system can be described using a number of methods from which it can be mentioned vector mechanics [1], [2], analytical mechanics [3] etc.

When vector mechanics is employed in studying the dynamics of rigid body, three theorems are available, two in vector form, namely the momentum theorem and the moment of momentum theorem and a scalar theorem, that is the kinetic energy theorem. It can be shown that, for rigid body dynamics, [4], the kinetic energy theorem is a consequence of the first two theorems.

Solving a dynamical problem using the Lagrange method, [5] assumes constructing a state function for the system, the kinetic energy. Using this state function, a number of scalar equations can be obtained and the number of independent degrees of freedom of the system should equal the number of scalar equations.

It is important to highlight that the approaching methods, both the vectorial one and the analytical one, assume knowing, besides other parameters, the parameters characteristic to the inertial properties of the system. Referring to the rigid body dynamics, these inertial characteristics are the mass and the inertia tensor. To determine the mass of a body may be a simple matter but finding the inertia matrix characteristic to a rigid body is a difficult task considering the fact that, due to tensorial characteristic, it modifies when the origin of axes changes and when the system of axes rotates.

II. THEORETICAL BACKGROUND

The concept of inertia tensor of a rigid is introduced when the moment of momentum of a rigid is defined:

$$\overline{\mathbf{K}}_0 = \int_M \overline{\mathbf{r}} \times (\mathbf{dm}\,\overline{\mathbf{v}}) = \int_M \overline{\mathbf{r}} \times (\overline{\boldsymbol{\omega}} \times \overline{\mathbf{r}})\mathbf{dm} \tag{1}$$

where, $\overline{\mathbf{r}}(\mathbf{x},\mathbf{y},\mathbf{z})$ is the position vector of the point where the elementary mass dm is positioned with respect to a point O attached to the rigid, M the mass of the rigid body , $\overline{\mathbf{v}}$ is the velocity of current point and $\overline{\boldsymbol{\omega}}$ is the angular velocity of the body. The double vector product from (1) can be written in an explicit manner, [6]:

$$\overline{\mathbf{r}} \times (\overline{\boldsymbol{\omega}} \times \overline{\mathbf{r}}) = \mathbf{r}^2\overline{\boldsymbol{\omega}} - (\overline{\boldsymbol{\omega}} \cdot \overline{\mathbf{r}})\overline{\mathbf{r}}\,. \tag{2}$$

Equation (2) can be expressed in matrix form as the product between a second order tensor and the column matrix associated to the $\overline{\boldsymbol{\omega}}$ vector:

$$\mathbf{r}^2\overline{\boldsymbol{\omega}} - \overline{\mathbf{r}}(\overline{\mathbf{r}}\overline{\boldsymbol{\omega}}) = \hat{\mathbf{I}}_3\mathbf{r}^2\overline{\boldsymbol{\omega}} - \overline{\mathbf{r}}\,\overline{\mathbf{r}} \cdot \overline{\boldsymbol{\omega}} = (\hat{\mathbf{I}}_3 - \overline{\mathbf{r}}\overline{\mathbf{r}})\overline{\boldsymbol{\omega}} \tag{3}$$

where $\hat{\mathbf{I}}_3$ is the unity matrix of third order and $\overline{\mathbf{r}}\,\overline{\mathbf{r}}$ is the dyadic product of vector $\overline{\mathbf{r}}$ with itself, [7]. Relation (1) can now be written under the form:

$$\begin{bmatrix} \mathbf{K}_{Ox} \\ \mathbf{K}_{Oy} \\ \mathbf{K}_{Oz} \end{bmatrix} = \left\{ \int_M \left[\hat{\mathbf{I}}_3\mathbf{r}^2 - \overline{\mathbf{r}}\overline{\mathbf{r}}\right]\mathbf{dm} \right\} \begin{bmatrix} \boldsymbol{\omega}_x \\ \boldsymbol{\omega}_y \\ \boldsymbol{\omega}_z \end{bmatrix}. \tag{4}$$

The matrix from (4) through which from angular velocity vector is obtained the matrix of kinetic momentum vector $\overline{\mathbf{K}}_0$ is the inertia tensor matrix. This matrix is denoted by $\hat{\mathbf{J}}_O$ and has the explicit form:

$$\hat{\mathbf{J}}_O = \int_M \begin{bmatrix} \mathbf{y}^2 + \mathbf{z}^2 & -\mathbf{xy} & -\mathbf{xz} \\ -\mathbf{xy} & \mathbf{z}^2 + \mathbf{x}^2 & -\mathbf{yz} \\ -\mathbf{xz} & -\mathbf{yz} & \mathbf{x}^2 + \mathbf{y}^2 \end{bmatrix} \mathbf{dm}\,. \tag{5}$$

By integration, it results that the inertia matrix is a symmetrical one, being précised by six elements: three elements on the diagonal, representing the axial moments of inertia:

$$\begin{aligned} J_x &= \int_M (y^2+z^2)dm \\ J_y &= \int_M (z^2+x^2)dm \\ J_y &= \int_M (x^2+y^2)dm \end{aligned} \qquad (6)$$

and three non-diagonal elements, representing the centrifugal moments:

$$J_{xy} = \int_M xydm;\ J_{xy} = \int_M xydm;\ J_{xy} = \int_M xydm \qquad (7)$$

When applying the moment of momentum theorem:

$$\frac{d\bar{K}_O}{dt} = \bar{M}_{ext} \qquad (8)$$

where $\bar{M}_{ext}$ represents the sum of external moments applied to the rigid body, expressed with respect to the point O, it must be considered the fact that the kinetic momentum is found with respect to a mobile reference system and the derivative is made in the fixed system and therefore the relationship between the absolute derivative and the relative derivative of a vector $\bar{U}$ is required, [4].

$$\frac{d\bar{U}}{dt} = \frac{\partial \bar{U}}{\partial t} + \bar{\omega} \times \bar{U}. \qquad (9)$$

Thus:

$$\frac{d\bar{K}_O}{dt} = \hat{J}_O \bar{\varepsilon} + \bar{\omega} \times (\hat{J}_O \omega), \qquad (10)$$

where $\bar{\varepsilon}$ represents the angular acceleration vector. The right member of (10) can be straightforwardly evaluated by writing in matrix form, using the anti-symmetrical matrix $\tilde{\omega}$ associated to the angular velocity vector:

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \qquad (11)$$

The moment of momentum theorem is written in matrix form as:

$$\begin{bmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{xy} & J_y & -J_{yz} \\ -J_{xz} & -J_{yz} & J_z \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} +$$
$$\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{xy} & J_y & -J_{yz} \\ -J_{xz} & -J_{yz} & J_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{bmatrix} \qquad (12)$$

Relation (12) can be applied for the case of a rigid body with fixed axis oscillating around a horizontal axis under its own weight, Fig. 1.

For the physical pendulum, Fig. 1, the Oz axis was chosen as joint axis and for the rotations it results: $\omega_x = \omega_y = 0$, $\varepsilon_x = \varepsilon_y = 0$, $M_{Ox} = M_{Oy}$, $M_z = -gd\sin\theta$, where g is the gravity acceleration, d is the distance from the joining point to the mass centrum and θ is the oscillation angle.

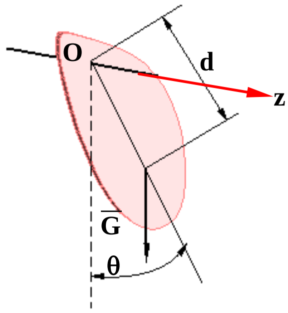

Fig. 1. Physical pendulum

$$\begin{bmatrix} -J_{xz}\varepsilon + J_{yz}\omega^2 \\ -J_{yz}\varepsilon - J_{yz}\omega^2 \\ J_z\varepsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -Mgd\sin\theta \end{bmatrix}. \qquad (13)$$

The pendulum motion is described by the equality of the last elements from the two members. Taking into account that

$$\varepsilon = d^2\theta / dt^2 = \ddot{\theta}. \qquad (14)$$

The motion equation of pendulum is:

$$J_{Oz}\ddot{\theta} + Mgd\sin\theta = 0. \qquad (15)$$

The equation (15) is an ordinary differential equation of second order and under this form a closed solution can not be found. With the hypothesis that the angular amplitude θ_{max} satisfies the condition:

$$\theta_{max} \le 5^o. \qquad (16)$$

The following approximation can be made:

$$\sin(\theta) \cong \theta\ (rad). \qquad (17)$$

Under the assumption (17), (15) becomes a homogenous differential equation with constant coefficients:

$$J_{Oz}\ddot{\theta} + Mgd\theta = 0. \qquad (18)$$

The pendulum motion will be a harmonic one, with the period:

$$T = 2\pi\sqrt{\frac{J_{Oz}}{Mgd}}. \qquad (19)$$

III. METHODOLOGY AND PROPOSED TEST RIG

The paper suggestion came from the Kater pendulum which is an instrument used for precise estimation of gravity acceleration, [8].

As mentioned, the purpose of the paper is to propose a method and a test rig used to find the axial moment of inertia of a solid. The method is based on finding the period of small free oscillations of a physical pendulum, using (19). To (19) is added the Steiner relation, which establishes the relationship between the axial moments J_G and J_O, found with respect to two parallel axes passing through point O and the mass centrum, respectively. By denoting with d the distance between the two axes, according to Steiner's theorem, [1], the following relation is written:

$$\mathbf{J_O = J_G + Md^2}. \tag{20}$$

Using (20) in (19) the next expression is obtained for the period of small oscillations:

$$\mathbf{T = 2\pi\sqrt{\frac{J_G + Md^2}{Mgd}}}\,. \tag{21}$$

The pendulum is presented in Fig. 2, having the two oscillation axes passing through points O_1 and O_2, at distances d_1 and d_2, respectively, about the centre of mass G. Applying (21) for the two suspending cases, the periods T_1 and T_2 are explicit and using additionally the obvious relation:

$$\mathbf{L = d_2 - d_1} \tag{22}$$

a system is obtained, with the unknowns the central moment of inertia J_G and the distances d_1, d_2 from the swinging points to the centre of mass.

$$\begin{cases} \mathbf{T_1 = 2\pi\sqrt{\frac{J_G + Md_1^2}{Mgd_1}}} \\ \mathbf{T_2 = 2\pi\sqrt{\frac{J_G + Md_2^2}{Mgd_2}}} \\ \mathbf{L = d_2 - d_1} \end{cases} \tag{23}$$

The solutions of the system can be written under the form:

$$\begin{aligned} \mathbf{d_1} &= \mathbf{L\frac{T_2^2 - T_0^2}{(T_2^2 - T_0^2) - (T_1^2 + T_0^2)}} \\ \mathbf{d_2} &= \mathbf{L\frac{T_1^2 + T_0^2}{(T_1^2 + T_0^2) - (T_2^2 - T_0^2)}} \\ \mathbf{J_G} &= \mathbf{\frac{ML^2}{T_0^2}\frac{T_0^2 - (T_2^2 - T_1^2)}{\frac{\left[(T_2^2 - T_0^2) - (T_1^2 + T_0^2)\right]^2}{(T_1^2 + T_0^2)(T_2^2 - T_0^2)}}} \end{aligned} \tag{24}$$

where $\mathbf{T_0}$ represents the period of the mathematical pendulum, having the $\mathbf{L}$ length:

$$\mathbf{T_0 = 2\pi\sqrt{\frac{L}{g}}}\,. \tag{25}$$

In order to confirm the method's precision, it was applied for finding the moment of inertia for a body with simple geometry, for which there are expressions for the calculus of central moment of inertia. Thus, an aluminium cylinder was considered, having the length 0.68(m) and the diameter 0.01(m). For a rod, the value of the moment of inertia about a normal axis passing through the centre of gravity is:

$$\mathbf{J_G = M\ell^2 / 12} \tag{26}$$

and for the used bar resulted $0.557 \cdot 10^{-3}\,(\mathrm{kg \cdot m^2})$.

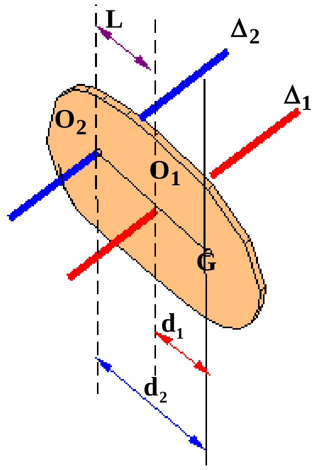

Fig. 2. Physical pendulum and the two oscillation axes

For finding the oscillation periods, a ring with two roller bearings was attached to the rod, as shown in Fig. 3. The ring can be fastened on the rod in different positions. The period of oscillation was found using an ultrasound non-contact sensor.

Fig. 3. System used for finding the period of oscillation

The results obtained for estimating the pendulum oscillation period are presented in Fig. 4. To increase the measurement accuracy, the time interval corresponding to five periods of oscillations was considered. On the rod, the marks where the ring was attached are visible and there were measured the following dimensions: L=0.205(m), d_1=0.334(m) and d_2=0.130(m).

The method was also applied for a device from the laboratory, a hammer pendulum. In Fig. 5 and Fig. 6 there are presented the CAD model and laboratory hammer, respectively, for which the axial momentum of inertia was found.

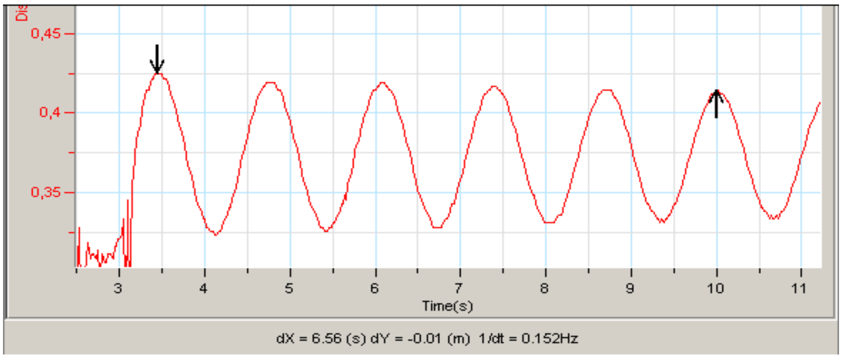

Fig. 4. Estimating the period of oscillation

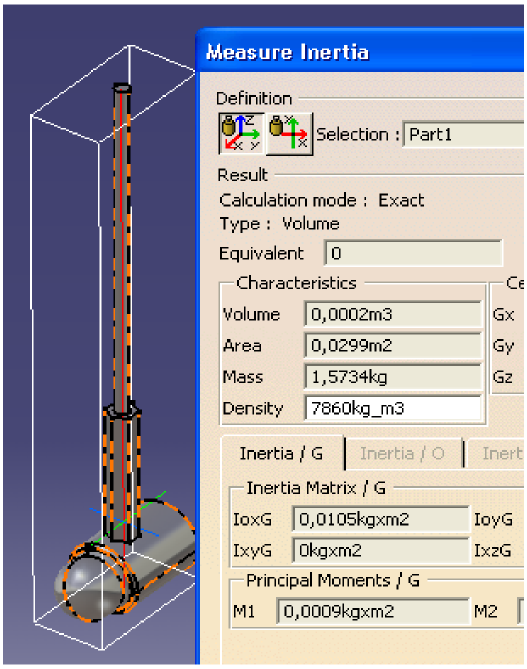

Fig. 5. The axial moment of inertia for a hammer pendulum obtained by CAD model

With the data found experimentally for T_1 and T_2, the solutions of system (24) were applied and the following values were obtained: d_1=0.099(m), d_2=0.303(m) and $J_G=10.9\cdot10^{-3}$(kg·m^2).

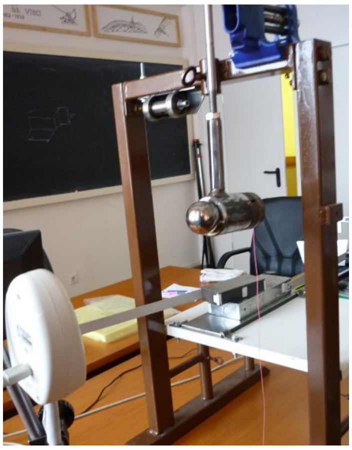

Fig. 6. The axial moment of inertia for a hammer pendulum obtained using the present method

The CAD modeling provided the axial moment of inertia as shown in Fig. 5, and the experimental values found with the present method are practically the same. In conclusion, the results are in very good agreement with the theoretical ones. The method's accuracy increases together with augmenting the distance between the swinging points.

IV. Conclusions

The paper proposes a methodology and device for finding the axial moment of inertia for a rigid body. The method is based on measuring the oscillation period of the body about two axes, knowing the distance between these axes. Following the method, the moment of inertia about a centroidal axis and the centre of mass of the body about the two axes are found. Using these data, the moment of inertia about any axis parallel to the initial axes can be found. The precision of the method was validated by applying it to a simple geometry body for which the analytical results can be compared to the experimental ones and a very good agreement was obtained.

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