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EQUIVALENT LINEARIZATION TECHNIQUES OF NON-LINEAR ROTORS

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Abstract—In this paper I present a new method for linearization of non-linear model of rotor - bearing systems. The model consists of a rotor treated as a continuous elastic shaft with several rigid disks, supported on the bearings with a non-linear behavior. The linearization method is based by optimizations principle. The linearization process by optimization methods, optimal linearization or equivalent linearization, means the replacing of the nonlinear system by an equivalent linear system. The equivalent linear system is obtained by finding of an equivalent stiffness matrix. The elements of the equivalent stiffness matrix are obtained by minimizing the cost functional defined as an error between the numerical solution of the nonlinear system and the linear system solution.

Keywords—finite element, non-linear bearing, non-linear programming, rotor dynamics, vibrations.

I. INTRODUCTION

THE non-linear phenomena are quite common in rotating machinery. The most significant reason for the non-linear phenomena are the fluid flow in hydrodynamic bearings, hydrostatic bearings, seals etc.. The dynamics of a rotor system supported by fluid film bearings is inherently a non-linear problem.

Non-linear elastic and energy dissipation properties of subsystem such as contact seals and elements with clearances such as partially wrong ball bearings are other sources for non-linear phenomena in rotor dynamics.

In the treatment of rotor-bearing system dynamic phenomena, was highlighted method for analysis of the highly nonlinear forces of fluid film bearings in combination with a continuous rotor model.

Many methods for vibration analysis of rotor systems have been developed: discretization methods, such as FEM, Transfer Matrix Method, and analytical methods [1]. Lee et al. in [2], use a modal analysis technique.

The dynamic analysis of non-linear rotating systems has been the subject of a number of studies, for example, those of Yamanchi [3] and Kim and Noah [4], who employed the method of harmonic balance. Dynamic condensation techniques were used with the harmonic balance method by Kim and Noah [4] to reduce the size of the system models.

Choi and Noah [5] added discrete Fourier transform procedures to the harmonic balance method and also included subharmonic response components.

Both linearized methods and non-linear approaches have been used in the modeling and solving rotor dynamic problems. Linearized models are commonly used in predictions of critical speeds, vibration response and instability threshold in a large range of operating points.

Non-linear models are used not only to verify results obtained from linearized model, but also to study some important rotor-bearing dynamic phenomena. Lund [6] used a linearized model to calculate the threshold speed of instability and damped critical speeds of a flexible rotor supported by journal bearings. Adams [7] used a non-linear model to simulate the response of a multi-bearing rotor system.

The present paper deals with a new linearization techniques of non-linear rotor-bearing system. The goal of the linearization method is obtaining a linear system that approximates the behaviour of the non-linear systems with sufficient accuracy of its equilibrium.

In the paper two strategies for vibration analysis of non-linear rotor-bearing systems that contain non-linear sub-systems are presented:

(I) direct integration of the non-linear differential equations in the time domain;

(II) linearization of the non-linear model using the optimization principle.

Computation Wilson-θ condensation techniques [8], in conjunction with an iteration procedure is used for obtain the numerical solution of non-linear model.

The author elaborated several computer codes in MATLAB programming language.

The non-linear programming problem is solved by using the method BFGS, Broyden-Fletcher-Goldfarb-Shanno.

II. NON-LINEAR MODEL OF ROTOR- BEARING SYSTEM

A. Uniform Shaft Element

The model consists of a rotor treated as a continuous elastic shaft with several rigid disks, supported on the bearings with a non-linear behavior.

In the study of the lateral motion of the rotor, the displacement of any point is defined by two translations (v, w) and two rotations (φ_y , φ_z) [7], [8].

The model use the beam C^1 finite element type based on Timoshenko beam model. The beam finite element has two nodes. In the case of the dynamic analysis four degrees of freedom (DOF) per node are considered: two displacements and two slopes measured in two perpendicular planes containing the beam [8].

The nodal displacements in the x-y and x-z planes, and corresponding vector of nodal forces are defined by two 4x1 sub-vector, respectively,

$$\{q_y^a\}=\begin{Bmatrix} v_i \\ \varphi_{zi} \\ v_j \\ \varphi_{zj} \end{Bmatrix}, \qquad \{q_z^a\}=\begin{Bmatrix} w_i \\ -\varphi_{yi} \\ w_j \\ -\varphi_{yj} \end{Bmatrix} \tag{1}$$

$$\{f_y^a\}=\begin{Bmatrix} T_{yi} \\ M_{zi} \\ T_{yj} \\ M_{zj} \end{Bmatrix}, \quad \{f_z^a\}=\begin{Bmatrix} T_{zi} \\ -M_{yi} \\ T_{zj} \\ -M_{yj} \end{Bmatrix} \tag{2}$$

B. Bearings

In the rotor system, the fluid film bearings play an important role in the dynamic behaviour of the system. In the case of linearized model of rotor bearing-systems, the physical model of fluid film bearing may be simplified as a linear element [8], [9]. The linearized bearing are commonly modelled as four spring coefficients and four damping coefficients, i.e.,

$$\begin{bmatrix} \mathbf{k}_{yy}^b & \mathbf{k}_{yz}^b \\ \mathbf{k}_{zy}^b & \mathbf{k}_{zz}^b \end{bmatrix}\begin{Bmatrix} \mathbf{q}_y^b \\ \mathbf{q}_z^b \end{Bmatrix}+\begin{bmatrix} \mathbf{c}_{yy}^b & \mathbf{c}_{yz}^b \\ \mathbf{c}_{zy}^b & \mathbf{c}_{zz}^b \end{bmatrix}\begin{Bmatrix} \dot{\mathbf{q}}_y^b \\ \dot{\mathbf{q}}_z^b \end{Bmatrix}=\begin{Bmatrix} \mathbf{f}_y^b \\ \mathbf{f}_z^b \end{Bmatrix} \tag{3}$$

$$[\mathbf{k}_{ij}^b]=\begin{bmatrix} k_{ij}^b & 0 \\ 0 & 0 \end{bmatrix}, \quad [\mathbf{c}_{ij}^b]=\begin{bmatrix} c_{ij}^b & 0 \\ 0 & 0 \end{bmatrix} \qquad i,j=y,z \tag{4}$$

It is well known that the behavior of both lubricated journal bearings and rolling element bearings is strongly non-linear and can cause rotors to behave in a non-linear way.

In this paper the nonlinearities are involved only in the elastic part of the system. The non-linear bearings have a cubic non-linear term [9], where the force–displacement relation for a non-linear spring element can be written as a function of the complex displacement z by the law

$$f(z)=k(1+|z|^2)z \tag{5}$$

This law is particularly well suited for modeling some rolling element bearings, in particular preloaded angular contact bearings but has a more general application.

Equation (5) has been widely used in non-linear dynamics, starting on the work of Duffing.

For example, in the case of the orthotropic bearing with horizontal and vertical principal directions, the force-displacement relations can be written

$$\begin{Bmatrix} f_y \\ f_z \end{Bmatrix}=\begin{bmatrix} k_{yy}+\hat{k}v^2 & 0 \\ 0 & k_{zz}+\hat{k}w^2 \end{bmatrix}\begin{Bmatrix} v \\ w \end{Bmatrix}. \tag{6}$$

The linear part of (5) can be introduced directly into the stiffness matrix.

The global mass matrix and the damping matrix are the same as in the linearized model of the bearings, but the global stiffness matrix contains non-linear terms $k+\hat{k}u^2$ in nodal displacements, due to the stiffness matrix of the bearings

$$[k]^{nl}=\begin{bmatrix} [k_{yy}^{nl}] & | & [0] \\ --- & -- & --- \\ [0] & | & [k_{zz}^{nl}] \end{bmatrix} \tag{7}$$

where

$$[k_{yy}^{nl}]=\begin{bmatrix} k_{yy}+\hat{k}v^2 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} k_{yy} & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} \hat{k}v^2 & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

$$[k_{zz}^{nl}]=\begin{bmatrix} k_{zz}+\hat{k}w^2 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} k_{zz} & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} \hat{k}w^2 & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

Thus, the global stiffness matrix can be written as the sum of two matrices by isolating the non-linear part

$$[K(\{x\})]^{nl}=[K]+[\hat{K}(\{x\})]^{nl} \tag{10}$$

where the first matrix is the stiffness matrix of the structure which refers to the shaft and to the constant terms of the bearing stiffness and the second matrix appears due to the non-linearity of the bearings.

The equations of motions of anisotropic rotor-bearing systems which consist of a flexible non-uniform axisymmetric shaft, rigid disk and anisotropic bearings are obtained in second order form, by assembling the element matrices and may be written as

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K(\{x\})]^{nl}=\{F\}. \tag{11}$$

In (11), $\{x\}$ is the global displacement vector, whose upper half contains the nodal displacements in the y-x plane, while the lower half contains those in z-y plane, and where the global stiffness matrix is defined by (10). The positive definite matrix $[M]$ is mass (inertia) matrix, the skew symmetric matrix $[G]$ is gyroscopic matrix, and the non-symmetric matrices $[C]$ and $[K]^{nl}$ are called the damping and the stiffness matrices, respectively.

III. NUMERICAL SOLUTION

The numerical time response solution for the non-linear system is calculated using the Wilson-θ method in conjunction with an iteration procedure. From the Wilson-θ method the resulting equations for acceleration and velocity vector can be expressed as

$$\{\ddot{x}\}_{t+\theta\Delta t} = \frac{6}{\theta^2\Delta t^2}\left(\{x\}_{t+\theta\Delta t} - \{x\}_t\right) - \frac{6}{\theta\Delta t}\{\dot{x}\}_t - 2\{\ddot{x}\}_t$$

$$\{\dot{x}\}_{t+\theta\Delta t} = \frac{3}{\theta\Delta t}\left(\{x\}_{t+\theta\Delta t} - \{x\}_t\right) - 2\{\dot{x}\}_t - \frac{\theta\Delta t}{2}\{\ddot{x}\}_t \quad (12)$$

From the equation of motion (11) written at the time $t+\theta\Delta t$ by using (12) we obtain the non-linear algebraic system

$$\left[\tilde{K}\right]^{nl}\{x\}_{t+\theta\Delta t} = \left\{\tilde{F}\right\} \quad (13)$$

with $\{x\}_{t+\theta\Delta t}$ as the unknown. In the (13) we used the following notations

$$\left[\tilde{K}\right] = [K] + \frac{6}{(\theta\Delta t)^2}[M] + \frac{3}{\theta\Delta t}[C] \quad (14)$$

$$\begin{aligned}\left\{\tilde{F}\right\} &= \{F\}_t + \theta\left(\{F\}_{t+\Delta t} - \{F\}_t\right)\\ &+ [M]\left(\frac{6}{(\theta\Delta t)^2}\{x\}_t + \frac{6}{\theta\Delta t}\{\dot{x}\}_t + 2\{\ddot{x}\}_t\right)\\ &+ [C]\left(\frac{3}{\theta\Delta t}\{x\}_t + 2\{\dot{x}\}_t + \frac{\theta\Delta t}{2}\{\ddot{x}\}_t\right)\end{aligned} \quad (15)$$

In the non-linear rotor's case, the stiffness matrix $[\tilde{K}]$ in (13) has non-linear terms, which depend on the values of the elements in vector $\{x\}_{t+\theta\Delta t}$.

Equation (13) is a set of non-linear algebraic equations now. Therefore an iteration procedure is utilized in conjunction with the Wilson-θ method to find $\{x\}_{t+\theta\Delta t}$ and then the displacements $\{x\}_{t+\Delta t}$.

The following steps describe the numerical procedure:

1. At $t=0$ specify initial conditions $\{x\}_0$, $\{\dot{x}\}_0$.
2. From the (9) using the known initial conditions from step 1, $\{\ddot{x}\}_0$ is calculated as
$$\{\ddot{x}\}_0 = [M]^{-1}\left(\{F\}_{t=0} - [C]\{\dot{x}\}_0 - [K]^{nl}\{x\}_0\right). \quad (16)$$
3. $i = i+1$
4. For the time step *i* at the moment of time *t*, assume a displacement vector $\{x\}_t \equiv \{x\}_i = \{x\}_i^*$.
5. Calculate the non-linear terms of the stiffness matrix $[\tilde{K}]$ using values from the assumed displacement vector.
6. Calculate displacement vector $\{x\}_{t+\Delta t} \equiv \{x\}_{i+1}$ using
$$\{x\}_{t+\Delta t} = \{x\}_t + \Delta t\{\dot{x}\}_t + \frac{\Delta t^2}{6}\left(\{\ddot{x}\}_{t+\Delta t} + 2\{\dot{x}\}_t\right) \quad (17)$$
and the stiffness matrix from Step (5).
7. The vector $\{x\}_i^* := \left(\{x\}_i^* + \{x\}_{i+1}\right)/2$ is modified.
8. Compare $\{x\}_{i+1}$ with assumed displacement vector
$$\sqrt{\left(\{x\}_{i+1} - \{x\}_i^*\right)^T\left(\{x\}_{i+1} - \{x\}_i^*\right)} < tol \quad (18)$$
9. If the difference is not within specified tolerances (*tol*) use an average value of the assumed displacement vector from Step (7) and the calculated displacement vector from Step (6) for the new assumed vector and return to Step (4). If the difference is within tolerance then update the assumed vector for t_{i+1} and go to Step (3). Continue for $i = N$ iterations to obtain the steady state solution.

IV. Equivalent Linearization model

The linearization of the non-linear model using the optimization principle means the replacing of the non-linear system (11) by an equivalent linear system

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]_{eq}^{lin}\{x\} = \{F\}. \quad (19)$$

The equivalent linear system is obtained by finding of an equivalent stiffness matrix $[K]_{eq}^{lin}$ which replaces the non-linear stiffness matrix $[K]^{nl}$.

The elements of the equivalent stiffness matrix are obtained by minimizing the cost functional defined as an error between the numerical solution of the non-linear system and the linear system solution i.e., the equivalent linear system is found by solving the following optimization problem

$$\begin{aligned}&\min\sum_{i=1}^{N}\left(\{x(t)\}^{num} - \{x(t)^{lin}\}\right)^2\\ &k_{yy}^i \le k_{yy}^{lin} \le k_{yy}^u ; k_{zz}^i \le k_{zz}^{lin} \le k_{zz}^u\end{aligned} \quad (20)$$

The numerical solution of the non-linear equation (9) was found using the Wilson-θ method in conjunction with an iteration procedure, shown in section III.

The author elaborated several computer codes in MATLAB programming language.

The optimization problem is solved by using the method BFGS, Broyden-Fletcher-Goldfarb-Shanno.

V. Numerical Example and conclusion

For the rotor model represented in Fig. 1, we propose a numerical example with the value $\hat{k} = 10^{14}\,N/m^3$, and the rotor data given in Table I.

The model comprises of a continuous elastic shaft mounted on two nonlinear bearing. One disk is mounted on the shaft.

Timoshenko beam model is adopted and gyroscopic effect is taken into account.

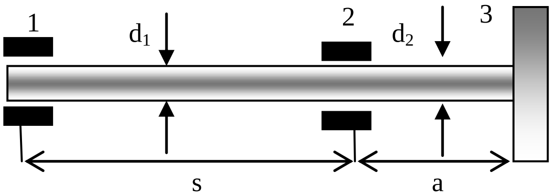

Fig.1. Rotor bearing system configuration

We shall determine an equivalent linear system for the optimized non-linear system, that means for the system with a determined distance between bearings by resolving an optimization problem [8]. The choice of objective functions depends on the type of the excitation: synchronous or asynchronous. The goal of the optimization is the determination of the design parameters so as the dynamic stiffness to be maximized.

In the case of non-linear rotors the optimization problem is formulated by the "energy" of the response as the objective function:

$$\min_{s} \frac{1}{T_{\max}} \int_{0}^{T_{\max}} \{u\}^T \{u\}\, dt \;,\; \{u\} = \begin{Bmatrix} v \\ w \end{Bmatrix} \qquad (21)$$

$$s^i \le s \le s^s$$

The optimization has been done for spin speed of the shaft $\Omega = 8000\,\text{rpm}$. There have been considered 60 rotations, *nrot* = 60, for the step number, *nstep* = 4096.

The variation of the cost functional with distance between bearing, both for the linear model $(\hat{k} = 0)$ and non-linear model, is represented in Fig. 2.

TABLE I
ROTOR DATA

Symbol	Quantity	Value
E	Young's modulus	E = 206,8 GPa
ρ	Density	ρ = 7833Kg/m^3
d	Shaft diameter	$d_1 = d_2$ = 0.08 m
s_{opt}	Optimal distance between bearings	s_{opt} = 0.35m
A	Initial distance between the right bearing and the disk	*a* = 0.1m
m	Disk mass	75 Kg
J_T	Transversal mass moment of inertia	J_T = 0.190 Kg m^2
J_P	Polar mass moment of inertia	J_T = 0.368 Kg m^2
e	Disk eccentricity	e = 0.01m
c_{yy}= c_{yy}	Damping coefficients	c_{yy}= c_{yy}=10^4 Ns/m
c_{zy}=c_{yz}	Damping coefficients	c_{zy}=c_{yz}= 0 Ns/m

The optimal value s_{opt}, of the distance *s* between the bearings obtain by solving optimization problem (21) is s_{opt} = 0,25 m.

Thus, numerical solution of non-linear systems $\{x(t)\}^{num}$ is calculated for this optimal configuration of the bearings.

The equivalent stiffness matrices, for station 1 and 2, obtained by solving of the non-linear programming problem (20) are:

- bearing 1

$$[k]_{eq}^{lin} = \begin{bmatrix} 6.0674\times10^8 & 0 \\ 0 & 3.9393\times10^8 \end{bmatrix} (N/m)$$

- bearing 2

$$[k]_{eq}^{lin} = \begin{bmatrix} 8.4330\times10^8 & 0 \\ 0 & 6.8278\times10^8 \end{bmatrix} (N/m)$$

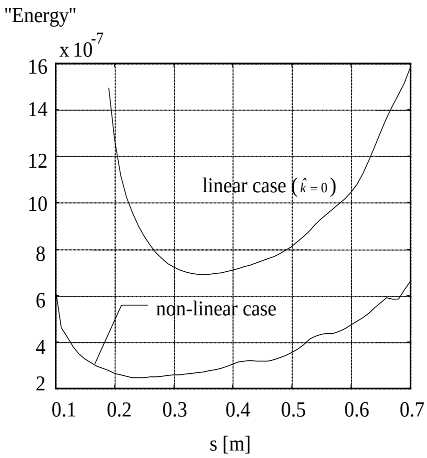

Fig. 2. The variation of the cost functional with distance between bearing

The optimal linearized model is easier to be used at:

- balancing of the multi-disk-rotor systems supported on bearings with non-linear flexibility;
- calculate the threshold speed of instability and damped critical speeds of a flexible rotor supported by journal bearings.

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