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OVERVIEW OF STRUCTURAL TOPOLOGY OPTIMIZATION METHODS FOR PLANE AND SOLID STRUCTURES

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***Abstract*—**Topology optimization for structural design is a special type of problem in the optimization field. Although there are efforts to apply classic optimization techniques, the particularities of topology structural optimization have given birth to dedicated, more efficient and reliable methods. This paper is intended as an overview of both the currently accepted and the promising new methods for topology optimization as found in the scientific literature of the last 20 years. Five types of methods are described and compared, highlighting the differences, advantages and pitfalls of each one: evolutionary-based algorithms, Solid Isotropic Microstructure with Penalization (SIMP) methods, Evolutionary Structural Optimization (ESO) methods, Soft-Kill Option (SKO) and level-set methods.

***Keywords*—**structural optimization, topology optimization, solid structures, overview

I. INTRODUCTION

STRUCTURAL optimization is a key element in the functional and technological design of load bearing structures. The engineer is posed with the difficult task of designing a structure by considering objectives that are often times contradictory, like minimizing total mass or volume, minimizing stress, maximizing stiffness, homogenizing stress distribution, ensuring proper manufacturability, minimizing production costs, etc. Structural optimization implies finding the optimum geometry with respect to one or more such criteria.

Structural optimization can be divided in three distinct branches, each targeting different types of parameters: topology, size and shape optimization. The techniques generally target either only topology or only size and shape optimization, with some rare exceptions that try to formulate the problem in a holistic way [1].

Topology optimization (TO) is the most general type of structural optimization, being performed in the initial phases of the design. All the feasible domain is considered, the aim being to find the most advantageous material distribution inside this domain, with respect to the design objectives. Topology optimization is responsible with most of the objective satisfaction (about 70% of the final design objective [2]), offering an initial model that can be fine-tuned afterwards with shape and size optimization methods.

A very important issue in structural optimization is the consideration of technological constraints, because a an optimized theoretical model that takes these aspects into account is much easier to be transformed in a viable final model, ensuring at the same time its quality of being "optimized", as opposed to a theoretical model that needs too many modifications to become manufacturable. If in size and shape optimization the technological constraints can be accounted for by adopting limits for the design parameters, in topology optimization the problem is more delicate, being necessary to alter the classic algorithms to encompass technological considerations, as in [1]-[4].

Some of the constraints when designing parts manufactured with classical technologies like casting or milling are: imposed direction(s) for cast mold sliding; connectivity control to avoid the checkerboard effect which produces volumes unconnected to the main structure; material influence radius, for a better material continuity inside the feasible domain; minimum or maximum thickness control, to ensure manufacturable dimensions, especially for cast parts; etc. All these limitations make the algorithms look for less optimized geometries with respect to mass, stress, stiffness or other objectives, but which are manufacturable. On the other hand, technological constraints can be much relaxed or even ignored in the case of rapid prototyping [5], [6], allowing the manufacturing of super-optimized structures not possible to achieve with other technologies.

II. TOPOLOGY OPTIMIZATION TECHNIQUES

As stated above, topology optimization is used in the initial phase of the design to obtain from the functional specifications the optimum material distribution inside the available volume of a structure. Also, it has the biggest optimization potential and thus a major influence on the behaviour of the final structure and its quality of

being "optimized" with respect to the design objectives.

On the other hand however, the results obtained by topology optimization can't be used directly, but need to be interpreted [2]. This procedure can be extremely difficult, especially in the case of volumetric structures, the designer needing to build models as close to the ones proposed by the optimization routines. If the interpretation of the results is not done properly, the whole optimization process loses its significance [7].

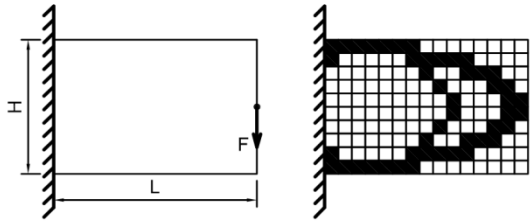


Fig. 1. Typical topology optimization problem.

The typical approach to plane or volumetric structures topology optimization is the discretization of the problem domain in a number finite elements and the assignment of full material, partial material or lack of material to each element, in a iterative scheme converging to the optimal material distribution inside the domain. Fig. 1 illustrates visually a possible result of a topology optimization problem for a cantilever beam modeled as a plane structure. The left picture represents the problem formulation and feasible domain, while the right one shows the finite element grid, with each element being either black (corresponding to a zone where material is needed) or white (corresponding to a zone where material is not necessary). A grid of only blacks and whites like the one in the picture is typical only to the so called "hard-kill" methods. However, many of the methods employ "soft-kill" techniques, which allow the finite elements to be "gray", corresponding to "fractional" material. These are usually needed to ensure proper convergence of the algorithm but are at least partially eliminated in the final iterations of the procedure.

The next sections offer a short survey of the most popular established and new methods for topology optimization found in the literature of the last 25 years.

A. Evolutionary based algorithms

The traditional approach to topology optimization with evolutionary algorithms (EA) is the discretization of the domain in a rectangular grid of square finite elements for the 2D problem (like in Fig. 1) or hexahedral elements for the 3D problem. Each element of this grid has a binary value attached to it [8], [9], 1 for the case when the element is filled with material (black square) and 0 for the case when the element represents a hole (white square).

Because the evolutionary algorithms are population based and the number of individuals forming the population needs to be of the same magnitude order as the number of optimization parameters (number of grid elements) for the algorithm to converge, applying these algorithms is extremely computationally expensive, especially for problems with a number of elements in the range of $10^4 - 10^6$ [26]. Indeed, in the case of 3D domains the problem needs tens or hundreds of thousands of fitness function evaluations, meaning as many FEA simulations, making the direct EA approach impossible.

To reduce the number of fitness evaluations, [10] proposes the generation of the optimal solution over a series of steps, each with an increasingly refined grid and with an increasingly longer chromosome. Each step starts from the best result of the previous step and the process continues until a sufficiently refined solution is obtained. This approach which manages to significantly reduce the number of FEA simulations is illustrated in Fig. 2.

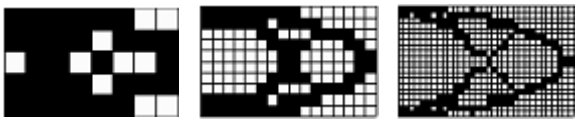


Fig. 2. Optimization with variable length chromosome [10].

Because of the discrete representation of the domain and the stochastic character of the evolutionary algorithms, the application of these in topology optimization leads to structure connectivity problems. To overcome these issues, many solutions have been proposed, among which: starting from seed elements (force application points, supports) and considering as filled with material only the elements connected to these [11]; using a chromosome mask to filter out unconnected elements; using a filter based on element compliance [10]; dynamic penalizing unconnected elements and image processing [12], etc. All these techniques are specifically efficient for plane structures, in the 3D case the problem becoming more complex and the discontinuity issues amplified.

a) **b)**

Fig. 3. TO with EA and a morphological representation of geometry; a) design domain; b) optimization result (CAD model) [13].

A novel and distinct approach to topology optimization with evolutionary algorithms, explored in papers such as [13] and [14], is to use a morphological representation of the geometry, instead the traditional discrete one. The basic primitives used to describe the geometry are spline curves or NURBS surfaces. In [13], the geometry is represented in a commercial CAD software, the optimization being carried out directly on

the CAD specification tree, each tree instance being a candidate solution. The genetic operators are applied to the structure and nodes of the specification tree, dynamically evolving the position and number of spline control points. Fig. 3 illustrates a sample result of this method. The feasible domain of a bracket (to the left), with supports at the four bottom corners and a force applied in the middle of the top face is found to have the optimum geometry represented at the right side of the figure. One of the great advantages of this approach is the fact that the result is a complete, final CAD model, as opposed to the classic TO methods that output results in a form that needs to be interpreted by the designer.

B. Solid Isotropic Microstructure with Penalization (SIMP)

SIMP is the most studied, implemented in commercial software and mathematically well-defined of all topology optimization methods. Starting from the base idea offered by the homogenization method [15], SIMP is proposed for the first time in [16] and its name coined in [17]. The method has been constantly developed and improved in books and papers such as [4], [18], [19].

SIMP is a "soft-kill" method, the design volume being divided into a grid of $\mathbf{N}$ elements (isotropic solid microstructures), each element $\mathbf{e}$ having a fractional material density $\boldsymbol{\rho}_e$. The objective function is the strain energy $\mathbf{SE}$, under a constraint of target volume $\mathbf{V}^*$, meaning the technique searches the material density distribution inside the design domain that minimizes strain energy for a preset structure volume. The densities of the microstructures are gathered in the vector $\mathbf{P}$ and represent the optimization parameters. Mathematically, the problem can be formulated as:

$$\mathbf{SE}(\mathbf{P}) = \sum_{\mathbf{e}=1}^{\mathbf{N}} (\boldsymbol{\rho}_\mathbf{e})^{\mathbf{p}} [\mathbf{u}_\mathbf{e}]^{\mathbf{T}} [\mathbf{k}_\mathbf{e}] [\mathbf{u}_\mathbf{e}] \tag{1}$$

subjected to the constraints:

$$\mathbf{V}^* - \sum_{\mathbf{e}=1}^{\mathbf{N}} \mathbf{V}_\mathbf{e} \boldsymbol{\rho}_\mathbf{e} = 0 \tag{2}$$

$$0 < \boldsymbol{\rho}_{\mathbf{min}} \le \boldsymbol{\rho}_\mathbf{e} \le 1 \tag{3}$$

In the equations above, $[\mathbf{u}_\mathbf{e}]$ represents the nodal displacement vector and $[\mathbf{k}_\mathbf{e}]$ the stiffness matrix of element $\mathbf{e}$. $\boldsymbol{\rho}_{\mathbf{min}}$ is the minimum allowable density (for empty elements), chosen greater than zero to ensure the stability of the finite elements analysis.

The penalty factor $\mathbf{p}$ is the main parameter of the method, its value being crucial to the algorithm behavior and success. The presence of p is needed to diminish the participation of fractional density (gray) elements to the total structural stiffness and to encourage in this way the development of elements which are either black ($\boldsymbol{\rho} = 1$) or white ($\boldsymbol{\rho} = \boldsymbol{\rho}_{\mathbf{min}}$). According to [1], $\mathbf{p}$ must be set in the interval $[2-4]$, [20] noting its value is usually $\mathbf{p} = 3$. As [26] suggests, to ensure the convergence and versatility of the method, $\mathbf{p}$ should be set to 1 and then gradually increased towards its final value.

In the case when technological limitations need to be considered, the algorithm must be modified by imposing supplementary constraints. For example, in the case of cast parts imposed mold sliding direction it is necessary for the elements on each line $\mathbf{k}$, parallel to the sliding direction, to have increasing densities along this line:

$$\left(0 \le \boldsymbol{\rho}_1 \le \boldsymbol{\rho}_2 \le \cdots \le \boldsymbol{\rho}_\mathbf{n} \le 1\right)_{\mathbf{k}=1..\mathbf{K}} \tag{4}$$

The basic SIMP algorithm is presented in Fig. 4. The starting point is a structure with all the elements having a density of 1 or with a random density distribution. This initial design is then iteratively evolved towards the optimal solution, each iteration assuming a number of steps and resulting in a new density distribution.

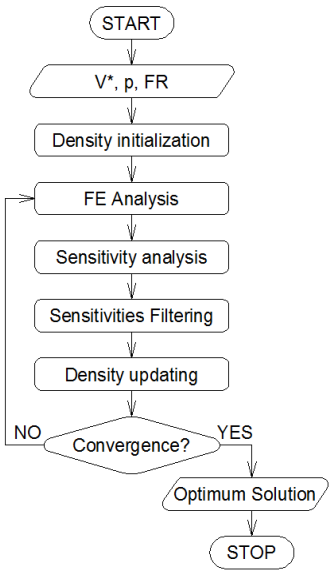


Fig. 4. Principle flow chart for the SIMP algorithm.

The first step in each iteration is a finite element analysis, considering current material densities. The results are used to evaluate the sensitivity of each element (the impact the variation of its density has on the objective function), expressed as the derivative of the objective function with respect to its density:

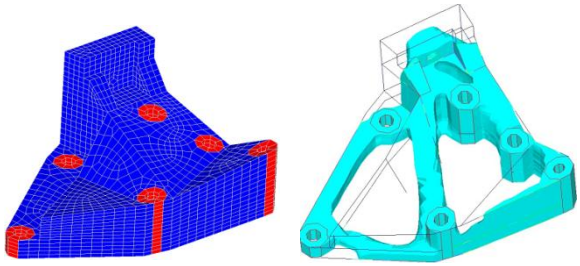
$$\frac{\partial \mathbf{SE}}{\partial \boldsymbol{\rho}_\mathbf{e}} = -\mathbf{p}(\boldsymbol{\rho}_\mathbf{e})^{\mathbf{p}-1} [\mathbf{u}_\mathbf{e}]^{\mathbf{T}} [\mathbf{k}_\mathbf{e}] [\mathbf{u}_\mathbf{e}] \tag{5}$$

Calculating each sensitivity independently and not considering any interaction between elements can lead to discontinuous structures, the so called "checkerboard effect". In order to reduce or even eliminate this problem, [19] proposes a filtering scheme for the sensitivities, by introducing an element filtering radius and averaging the sensitivities of each element considering the weighted influence of all the elements inside its influence sphere.

After applying the filtering scheme, these are used for

updating the sensitivities with the aim of finding a better behaving structure. One of the possibilities is the use of optimality criteria, according to which the objective function is modified to account for the constraints by introducing a Lagrange multiplier and each sensitivity is increased or decreased with a maximum allowed value, using these modified criteria.

To illustrate the possibilities offered by SIMP, Fig. 5 presents the result of topology optimization for a bracket, considering technological constraints.



a) b)

Fig. 5. TO with SIMP and imposed mold sliding direction; a) design space; b) optimized geometry [4].

C. Evolutionary Structural Optimization (ESO) methods

The methods in the ESO family are similar to SIMP in that they work with a discrete design space, but are "hard-kill" methods, meaning that each element in the domain has a density of either **0** (corresponding to a hole) or **1** (corresponding to material).

The first ESO was proposed in [21], the optimization starting with the whole design space filled with material and then eliminating iteratively inefficient elements. Subsequently, AESO ("additive evolutionary structural optimization") was proposed in [22], where the model starts from a structure which connects the seed points (load and supports) with a minimum number of elements and new elements are added iteratively around the elements with high sensitivity. To overcome the limitations of both ESO and AESO, a combination of the two was introduced in [23] and coined BESO ("bi-directional evolutionary structural optimization"). As the name implies, this method eliminates inefficient elements while at the same time adds new ones where needed. The initial BESO was later modified and enhanced to allow the consideration of 3D domains and multiple load cases [24]. At the same time, [25] proposes an enhanced, stable algorithm, independent of grid resolution and which ensures the convergence in most cases.

The optimization problem of BESO as formulated in [25] is posed in the form of minimization of mean compliance $\mathbf{C}$, under the target volume constraint $\mathbf{V}^*$.

$$\mathbf{C}(\mathbf{X}) = \frac{1}{2}[\mathbf{F}]^{\mathbf{T}}[\mathbf{u}] \tag{6}$$

subjected to the constraints:

$$\mathbf{V}^* - \sum_{\mathbf{e}=1}^{\mathbf{N}} \mathbf{V_e x_e} = 0\,,\ \mathbf{x_e} = \{0,1\} \tag{7}$$

In (6) and (7), $[\mathbf{F}]$ represents the vector of exterior forces, $[\mathbf{u}]$ the structure nodal displacement vector, $\mathbf{V_e}$ the volume of element $\mathbf{e}$, $\mathbf{x_e}$ the binary state of element $\mathbf{e}$ and $\mathbf{X}$ the vector containing all finite elements. The principle flow chart of the method as proposed in [25] is presented in Fig. 6.

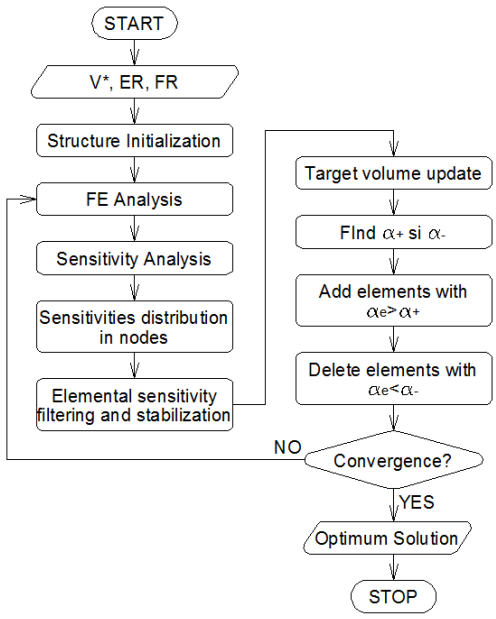


Fig. 6. Principle flow chart for the BESO algorithm.

Each iteration starts with a finite elements analysis of the current structure. Based on this analysis, a sensitivity for each element is calculated with the formula:

$$\boldsymbol{\alpha}_{\mathbf{e}} = \mathbf{SE}_{\boldsymbol{e}} = \frac{1}{2}[\mathbf{u_e}]^{\mathbf{T}}[\mathbf{k_e}][\mathbf{u_e}] \tag{8}$$

The sensitivity represents the quantity with which the total strain energy varies when the corresponding element is added to or removed from the structure. A high sensitivity means an important element which needs to be kept or added to the structure, having a high influence on the objective function.

As in SIMP, a filtering scheme is required to avoid "checkerboard" patterns. The filtering technique proposed in [25] first distributes the element sensitivities to the nodes, averaging for each node the sensitivities of the adjacent elements, weighted with the element volumes. The nodal sensitivities are then distributed back to the elements, averaging for each element the sensitivities of the nodes inside a sphere of preset radius ($\mathbf{FR}$), weighted with the distance from each node to the element's center of gravity. For a better convergence and stability of the algorithm, the final new sensitivities are averaged with the ones from the previous iteration.

After the computation and filtering of sensitivities, the current volume is modified (increased if smaller than $\mathbf{V}^*$, decreased if bigger, maintained constant if equal) with a prescribed fraction $\mathbf{ER}$, which represents the evolution

ratio of the volume towards the target. It is a crucial parameter of the algorithm, a small enough value being necessary to assure convergence and stability. If the convergence criterion is still not satisfied after the volume reaches $\mathbf{V}^*$, the algorithm continues without further modifying the volume, the number of added elements in each iteration being equal to the number of removed elements.

After the adjustment of the volume, the sensitivities are sorted in ascendant order and as function of how many elements need to be added and removed in order to reach the volume for the current iteration, the threshold sensitivities $\boldsymbol{\alpha}_-$ and $\boldsymbol{\alpha}_+$ are computed. All elements with $\boldsymbol{\alpha}_\mathbf{e} \le \boldsymbol{\alpha}_-$ are assigned a value of $\mathbf{x_e} = 0$, while all the elements with $\boldsymbol{\alpha}_\mathbf{e} \ge \boldsymbol{\alpha}_+$ are assigned a value of $\mathbf{x_e} = 1$. The algorithm is continued until the objective function reaches a stationary value over several generations.

D.Soft-Kill Option (SKO)

As the name implies, SKO [2], [3] is a soft-kill method, using a finite element grid and allowing the elements in the grid to have fractional material properties, just like the SIMP method. However, it also resembles the BESO technique, in that it iteratively adds and removes elements to the model on the basis of their stress state. Another difference to SIMP is the fact that it uses fractional elastic properties to represent how much material is needed in the grid elements, rather than fractional densities. As such, each element is assigned a Young modulus $\mathbf{E}$ in the interval $[\mathbf{E_{min}} - \mathbf{E_{max}}]$, depending on its temperature, which itself can be a value in the interval $[0-100]$ and is computed as a function of the element's stress state. The temperature has no physical significance, being just a convenient way of scaling material properties in FEA commercial programs.

A unique characteristic of SKO is the use of stresses as optimization objectives. The aim of the method is to find the geometry that gives a uniform distribution of the stresses, targeting a set reference stress $\boldsymbol{\sigma}_\mathbf{ref}$. To achieve this, each node in the grid has a temperature assigned to it $\mathbf{T_k^{(i)}}$, evaluated in each iteration i as a function of its values in the previous iteration $\mathbf{T_k^{(i-1)}}$ and the difference between the principal Von Mises stress at that point and the reference stress, scaled with a factor $\mathbf{s}$:

$$\mathbf{T_k^{(i)}} = \mathbf{T_k^{(i-1)}} + \mathbf{s}\left(\boldsymbol{\sigma}_\mathbf{k} - \boldsymbol{\sigma}_\mathbf{ref}\right) \tag{9}$$

This relation determines the stress of each element in the model to evolve towards the reference stress, with a speed proportional to the difference between current stress and reference stress. The end result is a structure with a uniform stress distribution. SKO can be modified to also consider technological constraints [3]. Fig. 7 shows an example: a bracket who's shaped is optimized using SKO and a mold sliding direction constraint.

a) **b)** **c)**

Fig. 7. SKO optimization with technological constraints; a) design domain and mold sliding direction; b) optimized model; c) final CAD model [3].

E. Level-set methods

The main idea behind level-set methods is the representation of the structure volume by means of an auxiliary continuous function ($\mathbf{F}$), with the number of variables equal to the number of spatial dimensions ($\mathbf{n}$). The optimization target becomes the function itself, instead of the actual design volume. Parameterizing a continuous function instead of an arbitrary domain eliminates the traditional difficulties in TO regarding material continuity. Unfeasible solutions are still possible, by the formation of continuous unconnected area, but these situations are much easier to tackle.

The optimization is performed by parameterizing and varying the parameters of the auxiliary function and by interpreting geometrically its values. This is done by considering the exterior boundary of the structure as the continuous set of points for which the function is null, set that can be expressed as a function $\boldsymbol{\Gamma}$ with $\mathbf{n}-1$ variables. In the 3D case, $\boldsymbol{\Gamma}$ is the spatial surface:

$$\boldsymbol{\Gamma} = \left\{(\mathbf{x},\mathbf{y},\mathbf{z}) \middle| \mathbf{F}(\mathbf{x},\mathbf{y},\mathbf{z}) = 0\right\} \tag{10}$$

The surface $\boldsymbol{\Gamma}$ delimits the volume for which $\mathbf{F}$ is positive, corresponding to the actual volume of the part. outside it, $\mathbf{F}$ is negative, corresponding to the interior and exterior zones with no material. As in the majority of TO methods, it is convenient to represent the problem domain as a grid of finite elements. Each finite element $\mathbf{i}$ is considered full (has material) if $\mathbf{F}$ is positive in its center of gravity:

$$\mathbf{F}(\mathbf{x_i},\mathbf{y_i},\mathbf{z_i}) \ge 0 \tag{11}$$

As highlighted in [26] TO optimization with level-set methods is extremely promising, but insufficiently studied yet. A complete review is offered in [27], where all the level-set methods proposed so far are presented, evaluated and compared.

III. CONCLUSIONS

Topology optimization is probably the most important type of structural optimization. Its use is crucial to give the designers an initial layout of the structure, optimized with respect to the functional and technological specifications. The most popular method, both in terms of research effort and implementation in commercial software is the SIMP method. BESO is another favorite

of the researchers, being employed in many papers, but less so in actual commercial implementations. However, BESO has great potential, especially considering the latest enhancements, especially when combined with other techniques like genetic algorithms. One issue that still needs to be addressed is the encompassment of technological constraints in the BESO methods.

Evolutionary algorithms, once posing great difficulties in topology optimization, have recently gain some ground with the combination with CAD software and a morphological representation of the geometry rather that a discrete one. This approach eliminates most of the classic disadvantages of EAs applied in TO.

SKO is a very interesting method, with obvious merits. It stands out as one of the few methods that targets the stresses inside the structure as the objective function. However, there isn't much research in this direction in the last 15 years, making the method a bit outdated. At the opposite end are the level-set methods, which represent a new set of TO techniques with great promise, but still lacking depth, generality and versatility.

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