ANNALS OF THE UNIVERSITY OF ORADEA
Fascicle of Management and Technological Engineering
ISSUE #3, DECEMBER 2014, http://www.imtuoradea.ro/auo.fmte/

DIFFERENTIAL FORMS

Gordana Jelić
University in Priština, Faculty of Technical Sciences, Kosovska Mitrovica, Serbia, e-mail: jelicgordana@ymail.com

***Abstract*—**In this paper, we have defined polylinear form and some specific polylinear forms such as symmetric and antisymmetric. The theorem that provides a condition when some p-form is antisymmetric. Differential p-form is defined, it is proved that it can unambiguously presented as

$\omega(x) = \sum \omega_{j1j2\ldots jp}(x)\ (\xi_{j1})\Lambda(\xi_{j2})\Lambda \ldots \Lambda(\xi_{jp})$, $1 \le j_1 < j_2 < \ldots j_p \le m$

Where $\omega j_1 j_{2\ldots} j_{1p}: D \to R$, D is an open set of space R^m. This differential for is n times differential when the function $\omega j_1 j_{2\ldots} j_{1p}$ is also n times differential. Then, the formula for the calculation of integral of the form ω on multiple M is given.

***Keywords*—**permutation, polylinear forms, external product, differential forms.

I. Antisymmetric Polylinear Forms

PERMUTATIONS: Let's say that J is a finite set of m elements. Each bijective reflection $\sigma: J \to J$ is called permutation of the set J.

Let's say that $b_1, b_2 \ldots, b_m$ are the elements of a set J and P_m is a number of its permutations. It is obvious that when m=1 then $P_1=1$. If m=2, then $P_2=2$: b_1, b_2; b_2, b_1. If m=3 then $P_3 = 6$: b_1, b_2, b_3; b_1, b_3, b_2; b_2, b_1, b_3; b_2, b_3, b_1; b_3, b_1, b_2; b_3, b_2, b_1.

By mathematical induction, it is shown that $P_m = m!$

If in permutation $m_1, m_2, \ldots m_m$ there is such a pair (m_i, m_j), where j>i and $m_j < m_i$ then we can say that pair forms INVERSION.

If the number of inversions is even, then permutation is called EVEN, if it is odd, the permutation is also called ODD.

Theorem 1: if two elements in permutation change places than the parity of permutation changes.

Proof: Let's assume that two adjacent elements of permutation change places. If the mappings $\sigma: J_m \to J_m$, $\tau: J_m \to J_m$ of permutation are such that

$\sigma(J_m) = \{m_1 \ldots m_{k-1}, m_k, m_{k+1}, m_{k+2} \ldots m_m\}$,
$\tau(J_m) = \{m_1, \ldots m_{k-1}, m_{k+1}, m_k, m_{k+2} \ldots m_m\}$

Pairs (m_i, m_j) in both cases simultaneously form or do not form inversion, if i<k and j>i are arbitrary and if j>k+1 and i<j are arbitrary.

One pair that does not satisfy the legality shown is (m_k, m_{k+1}) that transfers in (m_{k+1}, m_k) and therefore, number of inversions is changed for one, which means that parity is changed.

Let's assume now that two arbitrary elements change places, m_i and m_j. Then it is required to make $2\ i - j$ - 1 change of adjacent elements. If the number $2\ i - j$ - 1 is odd, then the parity of permutation is changed. This proves the theorem.

Let's take two permutations of the set J

$b_i' = \sigma(b_i)$, $b_i'' = \tau(b_i)$, $I = 1, 2, \ldots, m$.

Permutation products σ and τ is called the permutation $b_i''' = (\sigma \circ \tau)(b_i)$, $I = 1, 2, \ldots, m$ which is obtained firstly by the application of τ permutation and then σ permutation.

Permutation σ^{-1} is called REVERSE permutation σ, if $\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = I$ where I is identical permutation, i.e. $b_i = I(b_i)$, i = 1,2,…m.

If σ, τ and γ are three arbitrary permutations then $(\sigma \circ \tau) \circ \gamma = \sigma \circ (\tau \circ \gamma)$

In general case, product of permutation that is not a commutative operation i.e. $\sigma \circ \tau \neq \tau \circ \sigma$.

Permutation $i \to \sigma(i)$, $\sigma(i) = \sigma_i$, $i \in J_m$ is called TRANSPOSITION, if there is such a pair of different numbers $k \in J_m$, $\ell \in J_m$ that $k = \sigma_i$, $\ell = \sigma_k$ and $i = \sigma_i$ for any i different than k and ℓ.

If σ is the transposition, than σ^2 is identical to permutation.

Mapping

$G \to \{+1, -1\} : \sigma \to \varepsilon_\sigma =$
$+1, if\ \sigma\ even\ permutation$
$-1,\ if\ \sigma\ odd\ permutation$

That is based by SIGNATURE OF PERMUTATIONS.

II. Polylinear Anti-Synthetic Forms

Let's say that $E_1, E_2, \ldots E_p$ are vector spaces over the field R. Polylinear form is the mapping $\varphi: E_1 \times E_2 \times_{\ldots} \times E_p \to R$, so $\forall k \in 1, 2, \ldots, p$ is valid also for the fixed element system $a_i \in E_i$, $i \neq k$, function:

$X_k \to \varphi(a_1, \ldots, a_{k-1}, x_k, a_{k+1}, \ldots, a_p)$ meets the following conditions:

$\varphi(a_1, \ldots, a_{k-1}, \lambda x_k, a_{k+1}, \ldots, a_p) = \lambda\varphi(a_1, \ldots, a_{k-1}, x_k, a_{k+1}, \ldots, a_p)$
$\varphi(a_1, \ldots, a_{k-1}, x_{k'} + x_{k''}, a_{k+1}, \ldots a_p) = \varphi(a_1, \ldots a_{k-1}, x_{k'}, a_{k+1}, \ldots a_p) + \varphi(a_1, \ldots a_{k-1}, x_{k''}, a_{k+1}, \ldots a_p)$.

Polylinear form is anti-symmetric if it changes the sign in permutation of its two arguments.

Let's say that

$E_1=E_2=\ldots=E_p=R^m=E$, $R^m\times R^m\times\ldots\times R^m=E^p$,

Then polylinear form $\varphi: E^p\to R$ is called p-form or polylinear form of p degree. With the help of $\sigma:J_p\to J_p$ can be defined the mapping of $\sigma\varphi$:

$E^p\to R: \sigma\varphi\,(x_1, x_2,\ldots x_p) = \varphi\,(x_{\sigma 1}, x_{\sigma 2},\ldots x_{\sigma p})$

Polylinear form $\varphi:E^p\to R$ is called ANTI-SYNTHETIC, if $\sigma\varphi=\varepsilon_\sigma\,\varphi$, $\forall\,\sigma \in G_m$.

It follows that $\sigma\varphi=-\varphi$ if σ is transposition.

By symmetrization $S\varphi$ of the p-form of $\varphi:E^p\to R$, we imply p-form determined by the following equation

$S\varphi = \;{}_{\sigma\in G_p}\,\sigma\varphi.$

By anti-symmetrization $A\varphi$ of the p-form of $\varphi:E^p\to R$, we imply p-form determined by the equation

$A\varphi=\frac{1}{p!}\;{}_{\sigma\in G_p}\,\varepsilon_\sigma\sigma\varphi$

Theorem 2: let's say that $E=R_m$, $e_1, e_2\ldots e_m$ is the base of R_m space. Then, p-linear form $\varphi:E^p\to R$ is anti-symmetric, then and only than when it can be presented as follows

$\varphi\,(x_1, x_2,\ldots,x_p) = \sum \Delta_{j1,j2,\ldots j_p}\; a_{j1j2\ldots jp}$, $1\le j_1<j_2<\ldots<jp\le m$

where

$a_{j1j2\ldots j_p}\in R$, $\Delta_{j1,j2,\ldots j_p}=\det(x_{ijk})$, $1\le i\le p$, $1\le k\le p$.

One may see that last relation is unique and $aj_{1j2\ldots j_p}=\varphi(\ell_{j1j2\ldots j_p})$.

Elements $\Delta\, j_1, j_2\ldots j_p$ for the base of space $A_p\,(R^m, R)$ since the space $A_p\,(R^m, R)$ has a dimension $C_m^p = \;{}^{m}_{p}$.

Proof: let's say that $\varphi:E^p\to R$ is a p-linear anti-symmetric form. Then, if $e_1, e_2, \ldots e_m$ is the base of space R^m, then $x_i=\;{}^{m}_{j=1}\,x_{ij}e_j$, $i=1,2,\ \ldots\ ,p$, and due to polylinearity φ

$\varphi\ (x_1,x_2,\ \ldots\ ,x_p)\ =\ {}_{j'_1j'_2\ldots j'_p}\,x_{1j'_1}\,x_{2j'_2}\cdots x_{pj'_p}\quad\varphi\ (e_{1j'_1}\,e_{2j'_2}\cdots e_{pj'_p})$.

Let's take an arbitrary upward series of p indexes $j_1<j_2<\ldots j_p$ from the set J_n. Now, we will add all the summands within the sum III whose indexes $j_1', j_2'\ldots j_p'$ are permutations of $j_1, j_2\ldots j_p$. The amount of those summands is p!, where they make up a sum that can be presented as

${}_{\sigma\in G_p}\,\varepsilon_\sigma\;x1_{\sigma_1}\,x2_{\sigma_2}\cdots xp_{\sigma_p}\ \varphi\ (e_{1j_1}\,e_{2j_2}\cdots e_{pj_p})\ =\ \varphi\ (e_{1j_1}\,e_{2j_2}\cdots e_{pj_p})\,\det(x_{kjl})$, $1\le k\le p$, $1\le l\le p$.

By adding according to all growing indexes $j_1<j_2<\ldots<j_p$ from the set Jm, we obtain the sum I and equation II.

Now, we shall present the mapping of $\varphi:E^p\to R$ in the form I with arbitrary coefficients $a_{j1j2\ldots jp}$. Let's show that φ is anti-symmetric and p-linear.

Actually, φ is the sum of functions out of which each is in proportion with some determinant and that determinant is polylinear anti-symmetrization and, therefore, it represents an anti-symmetric form.

At the end, we need to mention that form I is unique, i.e. the equation II is certainly valid. Actually, if we put $x_i=\ell_{ki}$ for $i=1,2,\ldots p$ $(k_1<k_2<\ldots<k_p)$, then we get

$\Delta j_1,j_2\ldots j_p\,(\ell_{k1}, \ell_{k2}, \ldots \ell_{kp}) = \delta_{j1k1}\,\delta_{j2k2}\ldots\delta_{jpkp}$,

Where δ_{is} are Croneker symbols. From this it follows

$\varphi\,(\ell k1, \ell k2,\ldots \ell kp) = a_{k1,\,k2\ldots k_p}$,

i.e. coefficients II are found in one meaning form. This results in anti-symmetric p-forms $\Delta_{j1,\,j2\,\ldots j_p}$ form a base in space $A_p\,(R^m, R)$ and that dimension of that space is C_n^p. This proves the theorem.

Let's say that on R^m, p linear forms are given

$$\boldsymbol{\varphi_i : R^m \to R,\ i = 1,2,\ldots, p.} \qquad \textbf{(1)}$$

From these linear forms we can make a p-linear form

$$\boldsymbol{(x_1, x_2,\ldots,x_p) \to \varphi_1\,(x_1)\,\varphi_2\,(x_2)\,\ldots\,\varphi_p\,(x_p)} \qquad \textbf{(2)}$$

Anti-symmetization of the form (2)

$$\boldsymbol{(x_1, x_2,\ldots,x_p) \to \;{}_{\sigma\in G_p}\,\varepsilon_\sigma\ \varphi_1\,(x_{\sigma_1})\ \varphi_2\,(x_{\sigma_2})\ \ldots\ \varphi_p\,(x_{\sigma_p})} \qquad \textbf{(3)}$$

We call EXTERNAL PRODUCT of the form (1) and we label it in the following way $\varphi_1\wedge\varphi_2\wedge\ldots\wedge\varphi_p$.

Accordingly, for any vector system $x_1, x_2, \ldots, x_p$ from R^m the following equation is valid

$$\boldsymbol{(\varphi_1\Lambda\varphi_2\Lambda\ldots\Lambda\varphi_p)\,(x_1, x_2, \ldots, x_p) = \det\,(\varphi_i(x_j))} \qquad \textbf{(4)}$$

$1\le i\le p$

$1\le j\le p$

For two linear forms $x\to\varphi\,(x)$, $y\to\Psi\,(y)$, $x\in R^m$, $y\in R^m$, the following equation is valid

$(\varphi\wedge\Psi)\,(x,y) = \varphi(x)\,\Psi(y) - \varphi(y)\,\Psi(x)$.

Let's say that $\ell_1, \ell_2, \ldots \ell_m$ is a base of space R^m. We will mark with (ξ_i) the linear form, which joins each vector $x\in R^m$ its i coordinate. In that case, external product $(\xi_{i1})\wedge(\xi_{i2})\wedge\ldots\wedge(\xi_{ip})$ represents a p-linear anti-symmetric form, which according to (3) represents anti-symmetrization of p-form $((\xi_{i1})\wedge(\xi_{i2})\wedge\ldots\wedge(\xi_{jp}))\,(x_1, x_2, \ldots, x_p) = \det(x_{ijk})$, for: $1\le i\le p$ and $1\le k\le p$

Then, p-linear form, determined by form I can be written as

$$\boldsymbol{\varphi=\sum\ a_{j1j2\ldots j_p}\,(\xi_{j1})\,\Lambda\,(\xi_{j2})\,\Lambda\,\ldots\,\Lambda(\xi_{jp}),}$$
$$\boldsymbol{1\le j_1<j_2<\ldots<j_p\le m} \qquad \textbf{(5)}$$

III. DIFFERENTIAL FORMS

Let's say that D is an open set of the space R^m.

Definition: By differential form of degree p (or differential p-form) defined on D and whose values are in R, we imply the following function:
$\omega: D \rightarrow A_p(R^m, R)$.

Function ω mapps each point $x \in D$ in anti-symmetric p-form. Differential p-form is n times differentiable if the function ω is n times differentiable, where n is a positive number or + ∞.

Set of all n times differentiable p-forms on D with values in R we will mark with the symbol $\Omega_p^n(D,R)$. Set $\Omega_p^n(D,R)$ is vector space over the field R.

If $\omega \in \Omega_p^n(D,R)$, $x \in D$, $X_1, X_2, \ldots, X_p \in R^m$, then with $\omega(x)(X_1, X_2, \ldots, X_p) \in R$ we can mark the values of the function $\omega(x) \in A_p(R^m, R)$ on vector system $X_1, X_2, \ldots, X_p \in R^m$

Sometimes, those values are written with $\omega(x; X_1, X_2, \ldots, X_p)$.

Theorem 3: If $(\xi_i)$is a linear form that joins each vector $X \in R^m$ its i coordinate, then each differential p-form, determined on D with values in R, is presented as follows

$$\omega(z) = \sum_{1 \le j_1 < j_2 < \ldots < j_p \le m} \omega_{i1i2\ldots j_p}(z)\,(\xi_{j1}) \wedge (\xi_{j2}) \wedge \ldots \wedge (\xi_{jp}), \quad (6)$$

Where $\omega_{i1i2\ldots j_p}: D \rightarrow R$. this differential p-form is n times differentiable then and only then when the function $\omega_{i1i2\ldots j_p}$ is n times differentiable.

Proof: For each $x \in D$, the form ω(x) is anti-symmetric form. According to the theorem 2, each anti-symmetric p-form can be presented in the form of equation I. furthermore by using linear forms (ξ_i), each anti-symmetric p-form can be written in the form of equation (5) which matches the equation (6). Vector space $A_p(R^m, R)$ has a dimension C_m^p and it is identified with C_m^p product RxRx…Xr. Therefore, ω is n times differentiable then and only then when each of the components $\omega_{j1j2\ldots j_p}$ appears n times differentiable, where

$$D^k_\omega = \sum D^k \omega_{j1j2\ldots j_p}(\xi_{j1}) \wedge (\xi_{j2}) \wedge \ldots \wedge (\xi_{jp}),$$
$1 \le j_1 < j_2 < \ldots < j_p \le m$.

For partial extracts by coordinates, the following formula is valid:

$$\frac{\partial \omega}{\partial x_\alpha} = \sum \frac{\partial \omega}{\partial x_\alpha} \omega_{j1j2\ldots j_p}(\xi_{j1}) \wedge (\xi_{j2}) \wedge \ldots \wedge (\xi_{jp}), \quad 1 \le j_1 < j_2 < \ldots < j_p \le m.$$

The theorem is proved.

From now on, instead of (ξ_i), we will write dx_i where dx_i stands for linear form on R^m which joins i coordinate to each $X \in R^m$. Therefore, if $X = (X_1, X_2, \ldots, X_m)$, then
$dx_i(X) = X_i$
if $X_i = (X_{i1}, X_{i2}, \ldots, X_{im}) \subset R^m$, i=1,…, p then

$$dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p (X_1, X_2, \ldots, X_p) = \det(X_{ijk})_{\substack{1 \le i \le p \\ 1 \le k \le p}}$$

Now, differential p-form ω on $D \subset R^m$ with values in R has the following form:

$$\omega(x) = \sum_{1 \le j_1 < j_2 < \ldots < j_p \le m} \omega_{i1i2\ldots j_p}(x)\, dx_{j1} \wedge dx_{j2} \wedge \ldots \wedge dx_{jp}.$$

Right side of this equation is called canonical record of differential form. Its values on vector system $X_1, X_2, \ldots, X_p$ from R^m are determined according to the formula

$$\omega(x)(X_1, X_2, \ldots, X_p) = \sum_{1 \le j_1 < j_2 < \ldots < j_p \le m} \omega_{j1j2\ldots j_p}(x) \det(X_{ijk})_{\substack{1 \le i \le p \\ 1 \le k \le p}}.$$

Let's say that $\alpha \in \Omega_p^n(D,R)$, $\beta \in \Omega_q^n(D,R)$, $D \subset R^n$.

For each $x \in D$, α(x) belongs to the space $A_p(R^m, R)$ and β(x) to the space $A_q(R^m, R)$. Then, its external product is

$$\alpha(x) \wedge \beta(x) \in A_{p+q}(R^m, R).$$

By EXTERNAL PRODUCT of differential forms α and β we imply differential form $(\alpha \wedge \beta)(x) \in \Omega_{p+q}^n(D,R)$ where the mapping $x \rightarrow (\alpha \wedge \beta)(x)$ on vector system $X_1, X_2, \ldots, X_{p+q}$ from R^m is determined by

$$(\alpha \wedge \beta)(x; X_1, X_2, \ldots, X_{p+q}) = \sum_{\sigma \in G_{p+q}} \varepsilon_\sigma\, \alpha(x; X_{\sigma_1}, \ldots, X_{\sigma_p})\, \beta(x; X_{\sigma_{p+1}}, \ldots, X_{\sigma_{p+q}})$$

Where the summing is performed according to permutations of σ set J_{p+q} which meet the following condition $\sigma_1 < \sigma_2 < \ldots < \sigma_p$ and $\sigma_{p+1} < \sigma_{p+2} < \ldots < \sigma_{p+q}$.

Let's say that $f: D \rightarrow R$, $D \in R^m$ is differentiable function. In that case, its derivative $f' = \left(\frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \ldots \frac{\partial f}{\partial x_m}\right)$ determines differential form of the degree 1 (the first differential) on D with values in R. it is usually marked in the following way

$$d f(x) = \frac{\partial f}{\partial x_1}\, x\, dx_1 + \frac{\partial f}{\partial x_2}\, x\, dx_2 + \ldots + \frac{\partial f}{\partial x_m}\, x\, dx_m.$$

If there are p scalar differential functions given $f_i: D \rightarrow R$, $D \in R^m$, i=1,2,…p, then observing differentials as differential forms of degree 1, their external product is represented as

$$(d f_1 \wedge d f_2 \wedge \ldots \wedge d f_p)(x) = \sum \frac{D(f_1, f_2, \ldots, f_p)}{D(x_{j_1}, x_{j_2}, \ldots, x_{j_p})}\, d x_{j_1} \wedge d x_{j_2} \wedge \ldots \wedge d x_{j_p}.$$

Actually, if p=m, then

$$d f_1 \wedge d f_2 \wedge \ldots \wedge d f_m)(x) = \frac{D(f_1, f_2, \ldots, f_m)}{D(x_1, x_2, \ldots, x_m)}\, d x_1 \wedge d x_2 \wedge \ldots \wedge d x_m.$$

IV. INTEGRATION OF DIFFERENTIAL FORMS

Let's say that $M \subset R^n$ is p-dimensional compact multiplicity of the class C^1 and that orientation is given on M. Let's say that $\omega \in \Omega_p^0(U,R)$ is a differential p-form of the class C^0 in some environment U of M multiplicity. It is required to determine the integral

$$\int_M^{p} \omega$$

Of the p-form ω on multiplicity M, where (p) denotes the multiplicity of the integral.

Firstly, we shall observe one particular case, when the intersection of multiplicity M with carrier of p-form ω is contained in a related open set V ⊂ M, for which the parametrization of the class C1 is determined

φ : D→V

where D is related environment of null in R^p. Parametrization

$t \rightarrow \varphi(t)$, t ∈ D, $t = (t_1,\ldots t_p)$

we shall select in such a way that it is in accordance with given orientation M. it is obvious that M ∩ supp ω – compact is contained in V. therefore, its original φ^{-1} (M ∩ supp ω) – compact is contained in D. Let's observe differential form φ*ω, defined on a set D.

It can be written as

$f(t)\, dt_1 \wedge \ldots \wedge dt_p$.

If $\varphi(t) = (\varphi_1(t),\ldots,\varphi_n(t))$, t ∈ D, then we compliantly form the exchange of parameters

$$\varphi^*(\omega) = \sum \omega_{j_1\ldots j_p}\circ\varphi(t)\, \frac{D(\varphi_{j_1},\ldots,\varphi_{j_p})}{D(t_1,\ldots,t_p)}\, dt_1 \wedge \ldots \wedge dt_p,$$
$1 \le j_1 < \ldots < j_p \le n$,

And then

$$f(t) = \sum \omega_{j_1\ldots j_p}\circ\varphi(t)\, \frac{D(\varphi_{j_1},\ldots,\varphi_{j_p})}{D(t_1,\ldots,t_p)},\ 1 \le j_1 < \ldots < j_p \le n.$$

Function f is continuous on D with compact carrier.

Integral of the form ω on multiple M is determined by the equation

$$\int_M^{(p)} \omega = \int_D^{(p)} f(t)\, dt_1 \wedge \ldots \wedge dt_p = \int_D^{(p)} \varphi * \omega,$$

Or by using the form (1) we get

$$\int_M^{p} \omega = \int_D^{p} \omega_{j_1\ldots j_p}\circ\varphi(t)\, \frac{D(\varphi_{j_1},\ldots,\varphi_{j_p})}{D(t_1,\ldots,t_p)}\, dt_1 \wedge \ldots \wedge dt_p,$$
$$1 \le j_1 < \ldots < j_p \le n,$$

In that case, it follows

$$\int_D^{(p)} f(t)\, dt_1 \wedge \ldots \wedge dt_p = \theta \int_D^{(p)} f(t)\, dt_1 \ldots dt_p,$$

Where θ = +1 if the base R^p is positive and θ = -1 if the base R^p is positive. We need to prove that formula does not depend on the choice of parametrization φ.

Theorem 4: Let's assume that the intersection M ∩ supp ω of p-dimensional compact multiplicity M ⊂ R^n with the carrier of differential p-form ω of the class C^0 is contained in related open set V ⊂ M. In that case, the equation (2) is valid for any paramterization: D→V of the class C^1.

Proof: Let's assume that second parametrization ψ: D' → V' of the open set V' ⊂ M that contains compact M ∩ supp ω is given. Let's say that

$V_1 = V \cap V'$, $D_1 = \varphi^{-1}(V_1)$, $D_1 \subset D$, $D_1' = \psi^{-1}(V_1)$, $D_1' \subset D'$. If M ∩ supp ω is contained in the set V and in V', then (M ∩ supp ω) ⊂ D_1. Therefore, carrier of the form φ*ω is contained in D_1 and thus $\int_D^{(p)} \varphi * \omega = \int_{D_1}^{(p)} \varphi * \omega.$ (4)

For that reason,

$$\int_{D'}^{(p)} \boldsymbol{\psi * \omega} = \int_{D'_1}^{(p)} \boldsymbol{\psi * \omega}. \qquad \textbf{(5)}$$

If λ : $D_1' \rightarrow D_1$ some C* is differentiated and it retains the orientation, then the following equation $\psi = \varphi\circ\lambda$ is correct on the set D_1'.

From this, it follows that $\psi^*\omega = \lambda^*(\varphi^*\omega)$

In accordance to the theorem about the exchange of variables in p integral, we get

$$\int_{D'_1}^{(p)} \psi * \omega = \int_{D_1}^{(p)} \varphi * \omega.$$

From this, and from equations (4) and (5) we obtain

$$\int_{D'}^{(p)} \psi * \omega = \int_{D}^{(p)} \varphi * \omega.$$

Which proves the correctness of the definition of integral $\int_M^{(p)} \omega$ through the equation (2).

REFERENCES

[1] Damnjanović, Z., Mančić, D., Dašić, P., Lazarević, D., Pantović, R. Thermoelastic stress analysis based on infrared thermography. *Technics Technologies Education Management (TTEM)*, 7 (2), (2012), pp. 914-919.

[2] Jelić G.V., Stojanović V.S. Hopf invariant and its role in homotropics of mapping sphere to sphere. *Annals of the Oradea University - Fascicle of Management and Technological Engineering*, 12 (2), (2013), pp. 65-68.

[3] Jelić G.V., Stojanović V.S. Multiplicity as an object of study in contemporary differential geometry. *Annals of the Oradea University - Fascicle of Management and Technological Engineering*, 12 (2), (2013), pp. 69-72.

[4] Кудряцев, Л. Д. – Математическй анализ, Москва 1980.

[5] Lekić, M., Cvejić, S., Dašić, P. Iteration method for solving differential equations of second order oscillations. *Technics Technologies Education Management (TTEM)*, 7 (4), (2012), pp. 1751-1759.

[6] Lekić, M., Cvejić, S., Dašić, P. Oscillating two-amplitudinal solutions of the canonic differential equation of the second order. *Metalurgia International*, 18 (7), (2013), pp. 133-137.

[7] Ляшко И. И., Бояршук А. К., Гай Я. Г., Калайда А. Ф. - Математическй анализ, Киев 1985

[8] Расулов М. Л. – Применеия метода контурного интеграла, Москва 1980.